Matrices 1. Matrix Algebra

Matrix algebra.

Previously we calculated the determinants of square arrays of numbers. Such arrays are important in mathematics and its applications; they are called matrices. In general, they need not be square, only rectangular.

A rectangular array of numbers having \( m \) rows and \( n \) columns is called an \( m \times n \) matrix. The number in the \( i \)-th row and \( j \)-th column (where \( 1 \leq i \leq m, 1 \leq j \leq n \)) is called the \( ij \)-entry, and denoted \( a_{ij} \); the matrix itself is denoted by \( A \), or sometimes by \((a_{ij})\).

Two matrices of the same size are equal if corresponding entries are equal.

Two special kinds of matrices are the row-vectors: the \( 1 \times n \) matrices \((a_1, a_2, \ldots, a_n)\); and the column vectors: the \( m \times 1 \) matrices consisting of a column of \( m \) numbers.

From now on, row-vectors or column-vectors will be indicated by boldface small letters; when writing them by hand, put an arrow over the symbol.

Matrix operations

There are four basic operations which produce new matrices from old.

1. **Scalar multiplication**: Multiply each entry by \( c \) : \( cA = (ca_{ij}) \)

2. **Matrix addition**: Add the corresponding entries: \( A + B = (a_{ij} + b_{ij}) \); the two matrices must have the same number of rows and the same number of columns.

3. **Transposition**: The transpose of the \( m \times n \) matrix \( A \) is the \( n \times m \) matrix obtained by making the rows of \( A \) the columns of the new matrix. Common notations for the transpose are \( A^T \) and \( A' \); using the first we can write its definition as \( A^T = (a_{ji}) \).

If the matrix \( A \) is square, you can think of \( A^T \) as the matrix obtained by flipping \( A \) over around its main diagonal.

**Example 1.1** Let \( A = \begin{pmatrix} 2 & -3 \\ 0 & 1 \\ -1 & 2 \end{pmatrix} \), \( B = \begin{pmatrix} 1 & 5 \\ -2 & 3 \\ -1 & 0 \end{pmatrix} \). Find \( A + B, \ A^T, \ 2A - 3B \).

**Solution.** \( A + B = \begin{pmatrix} 3 & 2 \\ -2 & 4 \\ -2 & 2 \end{pmatrix} \); \( A^T = \begin{pmatrix} 2 & 0 & -1 \\ -3 & 1 & 2 \end{pmatrix} \);

\[
2A + (-3B) = \begin{pmatrix} 4 & -6 \\ 0 & 2 \\ -2 & 4 \end{pmatrix} + \begin{pmatrix} -3 & -15 \\ 6 & -9 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -21 \\ 6 & -7 \\ 3 & 0 \end{pmatrix}.
\]

4. **Matrix multiplication** This is the most important operation. Schematically, we have

\[
A \cdot B = C
\]

\( m \times n \quad n \times p \quad m \times p \)

\[
c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}
\]
The essential points are:

1. For the multiplication to be defined, \( A \) must have as many columns as \( B \) has rows;
2. The \( ij \)-th entry of the product matrix \( C \) is the dot product of the \( i \)-th row of \( A \) with the \( j \)-th column of \( B \).

**Example 1.2** \[
\begin{pmatrix}
2 & 1 & -1 \\
4 & 2 & -2 \\
2 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
-1 \\
2 \\
4
\end{pmatrix}
= ( -2 + 4 - 2 ) = ( 0 ) ;
\]

\[
\begin{pmatrix}
1 \\
2 \\
-1
\end{pmatrix}
\begin{pmatrix}
4 & 5 \\
8 & 10 \\
-4 & -5
\end{pmatrix}
= \begin{pmatrix}
2 & 0 & 1 \\
1 & -1 & -2 \\
1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & -1 \\
0 & 2 & 1 \\
-1 & 0 & 2
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 \\
3 & -2 & -6 \\
0 & 2 & 2
\end{pmatrix}
\]

The two most important types of multiplication, for multivariable calculus and differential equations, are:

1. \( AB \), where \( A \) and \( B \) are two square matrices of the same size — these can always be multiplied;
2. \( Ab \), where \( A \) is a square \( n \times n \) matrix, and \( b \) is a column \( n \)-vector.

**Laws and properties of matrix multiplication**

\textbf{M-1.} \( A(B + C) = AB + AC; \quad (A + B)C = AC + BC \) \quad \text{distributive laws}

\textbf{M-2.} \( (AB)C = A(BC); \quad (cA)B = c(AB) \). \quad \text{associative laws}

In both cases, the matrices must have compatible dimensions.

\textbf{M-3.} Let \( I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \); then \( AI = A \) and \( IA = A \) for any \( 3 \times 3 \) matrix.

\( I \) is called the \textit{identity} matrix of order 3. There is an analogously defined square identity matrix \( I_n \) of any order \( n \), obeying the same multiplication laws.

\textbf{M-4.} In general, for two square \( n \times n \) matrices \( A \) and \( B \), \( AB \neq BA \): \textit{matrix multiplication is not commutative}. (There are a few important exceptions, but they are very special — for example, the equality \( AI = IA \) where \( I \) is the identity matrix.)

\textbf{M-5.} For two square \( n \times n \) matrices \( A \) and \( B \), we have the \textit{determinant law}:

\[ |AB| = |A||B|, \quad \text{also written} \quad \det(AB) = \det(A)\det(B) \]

For \( 2 \times 2 \) matrices, this can be verified by direct calculation, but this naive method is unsuitable for larger matrices; it’s better to use some theory. We will simply assume it in these notes; we will also assume the other results above (of which only the associative law \textbf{M-2} offers any difficulty in the proof).

\textbf{M-6.} A useful fact is this: matrix multiplication can be used to pick out a row or column of a given matrix: you multiply by a simple row or column vector to do this. Two examples
should give the idea:

\[
\begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{pmatrix}
\begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix}
= \begin{pmatrix}
2 \\
5 \\
8
\end{pmatrix}
\]

the second column

\[
\begin{pmatrix}
1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{pmatrix}
= \begin{pmatrix}
1 & 2 & 3
\end{pmatrix}
\]

the first row