Logical Connectives

Mathematics works according to the laws of logic, which specify how to make valid deductions. In order to apply the laws of logic to mathematical statements, you need to understand their logical forms.

Proofs are composed of statements. A statement is a declarative sentence that can be either true or false.

Remark. Many real proofs contain things which aren’t really statements — questions, descriptions, and so on. They’re there to help to explain things for the reader. When I say “Proofs are composed of statements”, I’m referring to the actual mathematical content with the explanatory material removed.

Example. “Calvin Butterball is a math major” is a statement. You’d need to know more about Calvin and math majors to know whether the statement is true or false.

“0 = 1” is a statement which is false (assuming that “0” and “1” refer to the real numbers 0 and 1).

“Do you have a pork barbecue sandwich?” is not a statement — it’s a question. Likewise, “Eat your vegetables!” is not a statement — it’s an imperative sentence, i.e. an order to do something.

“1 + 1 = 2” is a statement. An easy way to tell is to read it and see if it’s a complete declarative sentence which is either true or false. This statement would read (in words):

“One plus one equals two.”

You can see that it’s a complete declarative sentence (and it happens to be a true statement about real numbers).

On the other hand, “1 + 1” is not a statement. It would be read “One plus one”, which is not a sentence since it doesn’t have a verb. (Things like “1 + 1” are referred to as terms or expressions.)

Since proofs are composed of statements, you should never have isolated phrases (like 1+1 or “(a+b)^2”) in your proofs. Be sure that every line of a proof is a statement. Read each line to yourself to be sure.

In terms of logical form, statements are built from simpler statements using logical connectives. The basic connectives of sentential logic are:

(a) Negation (“not”), denoted ∼.
(b) Conjunction (“and”), denoted ∧.
(c) Disjunction (“or”), denoted ∨.
(d) Conditional (“if-then” or “implication”), denoted →.
(e) Biconditional (“if and only if” or “double implication”), denoted ↔.

Later I’ll discuss the quantifiers “for all” (denoted ∀) and “there exists” (denoted ∃).

Remark. You may see different symbols used by other people. For example, some people use ¬ for negation. And ⇒ is sometimes used for the conditional, in which case ⇔ is used for the biconditional.

Example. Represent the following statements using logical connectives.

(a) P or not Q.

\[ P \lor \sim Q \]
(b) If P and R, then Q.

\((P \land R) \rightarrow Q\) \[2\]

(c) P if and only if (Q and R).

\(P \leftrightarrow (Q \land R)\) \[3\]

(d) Not P and not Q.

\(\sim P \land \sim Q\) \[4\]

(e) It is not the case that if P, then Q.

\(\sim (P \rightarrow Q)\) \[5\]

(f) If P and Q, then R or S.

\((P \land Q) \rightarrow (R \lor S)\) \[6\]

Other words or phrases may occur in statements. Here’s a list of some of them and how they are translated.

<table>
<thead>
<tr>
<th>Phrase</th>
<th>Logical translation</th>
</tr>
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<tbody>
<tr>
<td>P, but Q</td>
<td>(P \land Q)</td>
</tr>
<tr>
<td>Either P or Q</td>
<td>(P \lor Q)</td>
</tr>
<tr>
<td>P or Q, but not both</td>
<td>((P \lor Q) \land \sim (P \land Q))</td>
</tr>
<tr>
<td>P if Q</td>
<td>(Q \rightarrow P)</td>
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<td>P is sufficient for Q</td>
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<td>P only if Q</td>
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<td>P is equivalent to Q</td>
<td>(P \leftrightarrow Q)</td>
</tr>
<tr>
<td>P whenever Q</td>
<td>(Q \rightarrow P)</td>
</tr>
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</table>

Consider the word “but”, for example. If I say “Calvin is here, but Bonzo is there”, I mean that Calvin is here and Bonzo is there. My intention is that both of the statements should be true. That is the same as what I mean when I say “Calvin is here and Bonzo is there”.

In practice, mathematicians tend to a small set of phrases over and over. It doesn’t make for exciting reading, but it allows the reader to concentrate on the meaning of what is written. For example, a mathematician will usually say “if Q, then P”, rather than the logically equivalent “P whenever Q”. The second statement is less familiar, and therefore more apt to be misunderstood.

This is a good time to discuss the way the word “or” is used in mathematics. When you say “I’ll have dinner at MacDonald’s or at Pizza Hut”, you probably mean “or” in its exclusive sense: You’ll have dinner at MacDonald’s or you’ll have dinner at Pizza Hut, but not both. The “but not both” is what makes this an exclusive or.

Mathematicians use “or” in the inclusive sense. When “or” is used in this way, “I’ll have dinner at MacDonald’s or at Pizza Hut” means you’ll have dinner at MacDonald’s or you’ll have dinner at Pizza Hut, or possibly both. Obviously, I’m not guaranteeing that both will occur; I’m just not ruling it out.

**Example.** Translate the following statements into logical notation, using the following symbols:

\(S = \) “The stromboli is hot.”

\(L = \) “The lasagne is cold.”

\(P = \) “The pizza will be delivered.”

(a) “The stromboli is hot and the pizza will not be delivered.”

\(S \land \sim P\) \[4\]
(b) “If the lasagne is cold, then the pizza will be delivered.”

\[ L \rightarrow P \]

(c) “Either the lasagne is cold or the pizza won’t be delivered.”

\[ L \lor \sim P \]

(d) “If the pizza won’t be delivered, then both the stromboli is hot and the lasagne is cold.”

\[ \sim P \rightarrow (S \land L) \]

(e) “The lasagne isn’t cold if and only if the stromboli isn’t hot.”

\[ \sim L \leftrightarrow \sim S \]

(f) “The pizza will be delivered only if the lasagne is cold.”

\[ P \rightarrow L \]

(g) “The stromboli is hot and the lasagne isn’t cold, but the pizza will be delivered.”

\[ S \land \sim L \land P \]

The order of precedence of the logical connectives is:

1. Negation
2. Conjunction
3. Disjunction
4. Implication
5. Double implication

As usual, parentheses override the other precedence rules. In most cases, it’s best for the sake of clarity to use parentheses even if they aren’t required by the precedence rules. For example, it’s better to write

\[ (P \land Q) \lor R \] rather than \[ P \land Q \lor R. \]

Precedence would group \( P \) and \( Q \) anyway, but the first expression is clearer.

It’s not common practice to use parentheses for grouping in ordinary sentences. Therefore, when you’re translating logical statements into words, you may need to use certain expressions to indicate grouping.

- The combination “Either … or …” is used to indicate that everything between the “either” and the “or” is the first part of the “or” statement.
- The combination “Both … and …” is used to indicate that everything between the “both” and the “and” is the first part of the “and” statement.

In some cases, the only way to say something unambiguously is to be a bit wordy. Fortunately, mathematicians find ways to express themselves which are clear, yet avoid excessive linguistic complexity.
Example. Suppose that

\[ C = \text{"The cheesesteak is good."} \]
\[ F = \text{"The french fries are greasy."} \]
\[ W = \text{"The wings are spicy."} \]

Translate the following logical statements into words (with no logical symbols):

(a) \((\sim C \land F) \rightarrow W\)

"If the cheesesteak isn’t good and the french fries are greasy, then the wings are spicy.”

(b) \(\sim (C \lor W)\)

If I say “It’s not the case that the cheesesteak is good or the wings are spicy”, it might not be clear whether the negation applies only to “the cheesesteak is good” or to the disjunction “the cheesesteak is good or the wings are spicy”. So it’s better to say “It’s not the case that either the cheesesteak is good or the wings are spicy”, since the “either” implies that “the cheesesteak is good” or “the wings are spicy” are grouped together in the or-statement.

In this case, the “either” blocks the negation from applying to “the cheesesteak is good”, so the negation has to apply to the whole “or” statement.

(c) \(\sim (\sim W \land C)\)

"It’s not the case that both the wings aren’t spicy and the cheesesteak is good.” As with the use of the word “either” in (b), I’ve added the word “both” to indicate that the initial negation applies to the conjunction “the wings aren’t spicy and the cheesesteak is good”.

In this case, the “both” blocks the negation from applying to “the wings aren’t spicy”, so the negation has to apply to the whole “and” statement.

(d) \(\sim (\sim F)\).

The literal translation is “It’s not the case that the french fries aren’t greasy”. Or (more awkwardly) you could say “It’s not the case that it’s not the case that the french fries are greasy”.

Of course, this is logically the same as saying “The french fries are greasy”. But the question did not ask you to simplify the original statement — only to translate it, which you should do verbatim.

Example. Here are some examples of actual mathematical text.

(a) ([1], Theorem 25.11) In the semi-simple ring \(R\), let \(L = Re\) be a left ideal with generating idempotent \(e\). Then \(L\) is a minimal left ideal if and only if \(eRe\) is a skew field.

You could express this using logical connectives in the following way. Let

\[ A = \text{"}R\text{ is a semi-simple ring"}. \]
\[ B = \text{"}L = Re\text{ is a left ideal with generating idempotent } e\text{"}. \]
\[ C = \text{"}L\text{ is a minimal left ideal"}. \]
\[ D = \text{"}eRe\text{ is a skew field"}. \]

The statement can be translated as \((A \land B) \rightarrow (C \leftrightarrow D)\).

Notice that to determine the logical form, you don’t have to know what the words mean! Mathematicians use the word “let” to introduce hypotheses in the statement of a theorem. From the point of view of logical form, the statements that accompany “let” form the antecedent — the first part
of a conditional, as statements A and B do here. □

(b) ([2], Proposition 14.11) Let \( X \) and \( Y \) be CW-complexes. Then \( X \times Y \) (with the compactly generated topology) is a CW complex, and \( X \lor Y \) is a subcomplex.

Let

\[
P = "X \text{ and } Y \text{ are CW-complexes}"
\]

\[
Q = "X \times Y \text{ (with the compactly generated topology) is a CW complex}".
\]

\[
R = "X \lor Y \text{ is a subcomplex}".
\]

The proposition can then be written in logical notation as \( P \rightarrow (Q \land R) \).

Notice that you can often translate a statement in different ways. For example, I could have let

\[
A = "X \text{ is a CW-complex}".
\]

\[
B = "Y \text{ is a CW-complex}".
\]

\[
C = "X \times Y \text{ (with the compactly generated topology) is a CW complex}".
\]

\[
D = "X \lor Y \text{ is a subcomplex}".
\]

Now the proposition becomes \((A \land B) \rightarrow (C \land D)\). There is no difference in mathematical content, and no difference in terms of how you would prove it. □

As the last example shows, logical implications often arise in mathematical statements. Here’s some terminology. If \( P \rightarrow Q \) is an implication, then:

(a) \( P \) is the antecedent or hypothesis and \( Q \) is the consequent or conclusion.

(b) The converse is the conditional \( Q \rightarrow P \).

(c) The inverse is the conditional \( \sim P \rightarrow \sim Q \).

(d) The contrapositive is the conditional \( \sim Q \rightarrow \sim P \).

Example. Find the antecedent and the consequent of the following conditional statement:

“If \( x > y \), then \( y > \text{Calvin} \).”

Construct the converse, the inverse, and the contrapositive.

The antecedent is “\( x > y \)” and the consequent is “\( y > \text{Calvin} \)”.

The converse is “If \( y > \text{Calvin} \), then \( x > y \).”

The inverse is “If \( x \not> y \), then \( y \not> \text{Calvin} \).”

The contrapositive is “If \( y \not> \text{Calvin} \), then \( x \not> y \).”

Later on, I’ll show that a conditional statement and its contrapositive are logically equivalent. □

Example. Find the antecedent and the consequent of the following conditional statement:

“If Calvin gets a hot dog, then Calvin doesn’t get a soda.”

Construct the converse, the inverse, and the contrapositive.
The antecedent is “Calvin gets a hot dog” and the consequent is “Calvin doesn’t get a soda”.
The converse is “If Calvin doesn’t get a soda, then Calvin gets a hot dog”.
The inverse is “If Calvin doesn’t get a hot dog, then Calvin gets a soda”. (Note that the literal negation of the consequent is “It is not the case that Calvin doesn’t get a soda”. But the two negations cancel out — this is called double negation — so I get “Calvin gets a soda”.)
The contrapositive is “If Calvin gets a soda, then Calvin doesn’t get a hot dog”.  [4]

Different fields use different formats for citing sources. For instance, you may have seen books which referred to sources using footnotes. Mathematicians usually use numbers in square brackets (like “[1]” or “[2]”) for citations. The numbers refer to the references, which are listed at the end of the paper or book. Among other things, it makes for less clutter on the text pages, and is easier to typeset. Here are the references which I cited above.
