On Groups with Chain Conditions on Subnormal Subgroups

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Abstract: Groups with chain Conditions on subnormal subgroups have been investigated by many authors. In this paper we give a necessarily and sufficient conditions under which a group $G$ satisfy the ascending or the descending chain conditions on subnormal subgroups.

Keywords: Prime Subgroups, Groups with chain Conditions on subnormal subgroups.

1. INTRODUCTION

Let $G$ be a group. A subgroup $P$ of $G$ is said to be a primesubgroup of $G$ if $P$ is normal in $G$ and $[A,B] \subseteq P$ with $A,B < G$ implies that either $A \subseteq P$ or $B \subseteq P$. Here $[ , ]$ is the commutator. Following Scukin [11] we say that a group $G$ is prime if $[A,B] \neq 1$ whenever $A$ and $B$ are nontrivial normal subgroups of $G$, see also Dark [5]. Then $P$ is prime in $G$ if and only if $G/P$ is a prime group.

We define the soluble radical $\sigma(G)$ to be the product of all soluble normal subgroups of $G$. We say that $G$ is semisimple if $\sigma(G) = 1$. The terms of the derived and lower central series of $G$ are denoted $G^{(n)}$ and $\gamma_n(G)$ as in Robinson [8]. A prime subgroup $P$ of $G$ is said to be a minimal prime subgroup belonging to a normal subgroup $H$ if $P \supseteq H$ and if there is no prime subgroup between $H$ and $P$, see Kurata [6, p 205]. The radical $r(H)$ of a normal subgroup in $G$ is the intersection of all minimal prime subgroups belonging to $H$, see Kurata [6, p 206]. If $G$ is unclear we write this as $r(H)$. It follows that $r(H)$ is the intersection of all prime subgroups containing $H$, see Kurata [6, Proposition 1.13 p.207]. We write $r_G$. for $r_G(1).$, the intersection of all minimal prime subgroups of $G$.

We denote by $\text{Max} - \triangleleft$ the class of all groups satisfying the maximal condition on normal subgroups (often called Max-n), with similar definition for $\text{Min} - \triangleleft$, $\text{Max} - \triangleleft^n$, $\text{Min} - \triangleleft^n$. The classes of all groups satisfying the maximal (respectively minimal) condition on subnormal subgroups are denoted by Max-sn, and Min-sn, following Robinson[8], which is also our source for any other unexplained notation and determined by the corresponding chain condition, so that $G$ satisfies Min-sn and $G \in \text{Min} - sn$ are equivalent statement.

2. RESULT

Proposition 1: For all group $G$,

(a) $\sigma(G) \subseteq r_G$.

(b) If $G \in \text{Max} - \triangleleft$, then $\sigma(G) = r_G$.

Proof

(a) Let $H$ be a soluble normal subgroup of $G$. Then $H^{(n)} = 1$ for some $n \geq 0$. In particular $H^{(n)} \subseteq P$ for every prime subgroup $P$. Inductively we see that $H \subseteq P$, whence $\sigma(G) \subseteq r_G$. 

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Let \( R = r_G \) and suppose that \( R \) not soluble. Let \( C \) be the collection of all normal subgroups \( N \) of \( G \) such that \( R^{(n)} \not\subset N \) for all integers \( n \geq 0 \). Then \( C \) is non-empty since \( 1 \in C \). Hence \( C \) has a maximal element say \( p \). We claim that \( p \) is prime. Suppose not, then there are normal subgroups \( A, B \) of \( G \) such that \( A \not\subset p \) and \( B \not\subset p \) but \( [A, B] \subset p \). Therefore \( A, B \) is soluble. Hence \( R^{(n)} \subset AP \) and \( R^{(m)} \subset BP \) for some integers \( m, n \geq 0 \). Let \( s = \max \{m, n\} \). Then \( R^{(s+1)} \subset [AP, BP] = [AP, B][AP, P] = [A, B][P, B][A, P][P, P] \subset P \). Hence \( AP \subset P \) or \( BP \subset P \), which implies that \( A \subset P \) or \( B \subset P \), a contradiction. Hence \( p \) is prime and \( p \subset \sigma(G) \). But \( \sigma(G) \subset R \) by (a), so \( R = \sigma(G) \) as claimed.

**Proposition 2**

(a) Let \( G \in Max^– <1^3 \). Then \( \sigma(G) \) is soluble and \( \sigma(G) \in Max \).

(b) Let \( G \in Min^– <2^2 \). Then \( \sigma(G) \) is soluble and \( \sigma(G) \in Min \).

**Proof:**

(a) Since in particular \( G \in Max^– <1 \) it follows that \( S = \sigma(G) \) is the product of finitely many soluble normal subgroups, hence is soluble. Because \( G \in Max^– <3^3 \) we have \( S \in Max^– <2^2 \).

Each derived factor \( S^{(s)} \) is abelian with \( Max^– <1 \), hence with \( Max \). By E-closure of \( Max \), we have \( S \in Max^– <1 \).

(b) By Theorem 5.49.1 or Robinson [8] p. 148 we have \( Min^– <2^2 = Min^– sn \). Now apply the analogous argument to part (a) with Max replaced by Min.

**Proposition 3** Let \( G \) be a group and \( S \) be respectively the set of normal subgroups, subnormal subgroups, n-step subnormal subgroups of \( G \). Suppose that \( N_i < G \) (\( i = 1, \ldots, m \)) and \( \bigcap_{i=1}^{m} N_i = 1 \).

Let \( S = \{ HN_i \cap N_i : H \in S \} \in Max – S \) (respectively \( Min – S \)) for all \( I \),

then \( G \in Max – S \) (respectively \( Min – S \)).

**Proof:** This is equivalent to \( R_0 \)-closure of these classes, see Robinson[8] Corollary to Lemma 1.48, p.39.

**Proposition 4** A group \( G \) is a sub-direct product of a family of groups \( \{ G_{\alpha} \}_{\alpha \in A} \) if and only if for each \( \alpha \in A \) there is a surjective homomorphism \( g_{\alpha} : G \rightarrow G_{\alpha} \) such that \( \bigcap_{\alpha \in A} \ker g_{\alpha} = 1 \).

**Proof:** This is standard: compare Cohn [3, p.99]

**Corollary 5** Let \( G \) be a group and let \( \{ G_{\alpha} \}_{\alpha \in A} \) be a family of normal subgroups of \( G \).

If \( \bigcap_{\alpha \in A} G_{\alpha} = 1 \), then \( G \) is a sub-direct product of the family of groups \( \{ G_{\alpha} \}_{\alpha \in A} \).

**Proposition 6**

(a) \( G \) is semi-simple with \( Max^– <1^2 \) (respectively \( Max^– sn \)) if and only if \( G \) is a sub-direct product of a finite number of prime groups satisfying \( Max^– <1 \) (respectively \( Max^– sn \)).

(b) \( G \) is semi-simple with \( Min^– <2^1 \) (respectively \( Min^– sn \)) if and only if \( G \) is a sub-direct product of a finite number of prime groups satisfying \( Min^– <2 \) (respectively \( Min^– sn \)).
Proof: (a) Let $G$ be semisimple with $\text{Max} - \langle n \rangle$ (respectively $\text{Max} - \langle n \rangle$). Then $\sigma(G)$ = 1. By Kurata [3] Proposition 4p 214 we have $r_G = \bigcap_{i=1}^{m} P_i$ where the $P_i$ are minimal prime subgroups of $G$. But by Proposition 1(b) $\sigma(G) = r_G$, so $\sigma(G) = 1$. Since $P_i$ is a prime subgroup the quotient $G/P_i$ is prime, and by Q-closure it lies in $\text{Max} - \langle n \rangle$ (respectively $\text{Max} - \langle n \rangle$).

By Corollary 5 $G$ is a subdirect product of prime groups satisfying $\text{Max} - \langle n \rangle$ (respectively $\text{Max} - \langle n \rangle$).

To prove the converse suppose that $G$ is a subdirect product of finitely many prime groups $G_i$ where $i=1,\ldots,m$ and each $G_i$ satisfies $\text{Max} - \langle n \rangle$ (respectively $\text{Max} - \langle n \rangle$). Let $g_i : G \to G_i$ be the homomorphism of Proposition 4. For each $I$ we have $G/\ker g_i \cong G_i$, and $G_i$ is prime. So $\ker g_i$ is a prime subgroup of $G$. Thus $r_G \subseteq \ker g_i$ for all $i$, so $r_G = 1$. By proposition 1(a) also $\sigma(G) = 1$, so $G$ is semisimple. That $G \in \text{Max} - \langle n \rangle$ (respectively $\text{Max} - \langle n \rangle$) follows from Proposition 3.

(b) Let $G$ be semisimple with $\text{Min} - \langle n \rangle$ (respectively $\text{Min} - \langle n \rangle$). Then $G$ is has only a finite number of minimal normal subgroups where $i=1,\ldots,r$. Let $M_j$ be a normal subgroup of $G$ that is maximal with respect to not containing $M_j$. We claim that $P_i$ is a prime subgroup of $G$. If not there exist normal subgroups $A, B$ of $G$ such that $A \nsubseteq P_i$, $B \nsubseteq P_i$, but $[A, B] \subseteq P_i$. Now $P_i \subseteq AP_i$ and $P_i \subseteq BP_i$, so by the choice of $P_i$ we have $AP_i \supseteq M_i$ and $BP_i \supseteq M_j$. Therefore $\gamma_2 M_i \subseteq [AP_i, BP_i] \subsetneq P_i$. But $\gamma_2 M_i \neq 1$ since $G$ is semi-simple so $\gamma_2 M_i = M_i \subseteq P_i$. Therefore $P_i$ is a prime subgroup of $G$ and $G/P_i$ is a prime group. If $\bigcap_{i=1}^{m} P_i \neq 1$, then this intersection contains some minimal subgroup $M_j$. But $M_j \nsubseteq P_i$, a contradiction. Therefore $\bigcap_{i=1}^{m} P_i = 1$ and Corollary 5 implies that $G$ is a sub-direct product of a finite number of prime groups with $\text{Min} - \langle n \rangle$ (respectively $\text{Min} - \langle n \rangle$). The converse is as in part (a).

We now come to our main theorem:

**Theorem 7** Let $G$ be a group. Then

(a) $G \in \text{Max} - \langle n \rangle$ (respectively $\text{Max} - \langle n \rangle$) if and only if

(i) $\sigma(G)$ is soluble with $\text{Max}$. 

(ii) $G/\sigma(G)$ is a sub-direct product of finitely many prime groups satisfying

$\text{Max} - \langle n \rangle$ (respectively $\text{Max} - \langle n \rangle$)

(b) $G \in \text{Min} - \langle n \rangle$ (respectively (or equivalently $\text{Min} - \langle n \rangle$) if and only if

(i) $\sigma(G)$ is soluble with $\text{Min}$. 

(ii) $G/\sigma(G)$ is a sub-direct product of finitely many prime groups satisfying

$\text{Min} - \langle n \rangle$ (respectively $\text{Min} - \langle n \rangle$)

**Proof:** Combine Propositions 2 and 6.
Corollary 8: G is a finite group if and only if $\sigma(G)$ is finite and $G/\sigma(G)$ is a subdirect product of finitely many finite prime groups.

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