Mathematics

"Maths" and "Math" redirect here. For other uses of "Mathematics" or "Math", see Mathematics (disambiguation) and Math (disambiguation).

Euclid, Greek mathematician, 3rd century BC, as imagined by Raphael in this detail from The School of Athens.[1]

Mathematics is the study of quantity, structure, space, and change. Mathematicians seek out patterns, formulate new conjectures, and establish truth by rigorous deduction from appropriately chosen axioms and definitions.[2]

There is debate over whether mathematical objects such as numbers and points exist naturally or are human creations. The mathematician Benjamin Peirce called mathematics "the science that draws necessary conclusions".[5] Albert Einstein, on the other hand, stated that "as far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality."[6]

Through the use of abstraction and logical reasoning, mathematics evolved from counting, calculation, measurement, and the systematic study of the shapes and motions of physical objects. Practical mathematics has been a human activity for as far back as written records exist. Rigorous arguments first appeared in Greek mathematics, most notably in Euclid's Elements. Mathematics continued to develop, for example in China in 300 BCE, in India in 100 CE, and in Arabia in 800 CE, until the Renaissance, when mathematical innovations interacting with new scientific discoveries led to a rapid increase in the rate of mathematical discovery that continues to the present day.[7]

Mathematics is used throughout the world as an essential tool in many fields, including natural science, engineering, medicine, and the social sciences. Applied mathematics, the branch of mathematics concerned with application of mathematical knowledge to other fields, inspires and makes use of new mathematical discoveries and sometimes leads to the development of entirely new mathematical disciplines, such as statistics and game theory. Mathematicians also engage in pure mathematics, or mathematics for its own sake, without having any application in mind, although practical applications for what began as pure mathematics are often discovered.

Etymology

The word "mathematics" comes from the Greek μάθημα (máthēma), which means learning, study, science, and additionally came to have the narrower and more technical meaning "mathematical study", even in Classical times.[9] Its adjective is μαθηματικός (mathēmatikós), related to learning, or studious, which likewise further came to mean mathematical. In particular, μαθηματική τέχνη (mathēmatikē tékhνē), in Latin ars mathematica, meant the mathematical art.

The apparent plural form in English, like the French plural form les mathématiques (and the less commonly used singular derivative la mathématique), goes back to the Latin neuter plural mathematica (Cicero), based on the Greek plural τα μαθηματικά (ta mathēmatiká), used by Aristotle, and meaning roughly "all things mathematical"; although it is plausible that English borrowed only the adjective mathematic(al) and formed the noun mathematics anew, after the pattern of physics and metaphysics, which were inherited from the
In English, the noun *mathematics* takes singular verb forms. It is often shortened to *math* in English-speaking North America, or *maths* in other English-speaking regions.

**History**

*Main article: History of mathematics*

![A quipu, used by the Inca to record numbers.](image)

The evolution of mathematics might be seen as an ever-increasing series of *abstractions*, or alternatively an expansion of subject matter. The first abstraction, which is shared by many animals,[11] was probably that of **numbers**: the realization that two apples and two oranges (for example) have something in common.

In addition to recognizing how to **count** physical objects, **prehistoric** peoples also recognized how to count **abstract** quantities, like **time** – days, seasons, years.[12] **Elementary arithmetic** (**addition**, **subtraction**, **multiplication** and **division**) naturally followed.

Further steps needed **writing** or some other system for recording numbers such as **tallies** or the knotted strings called *quipu* used by the **Inca** to store numerical data. [citation needed] **Numeral systems** have been many and diverse, with the first known written numerals created by **Egyptians** in **Middle Kingdom** texts such as the Rhind Mathematical Papyrus.[citation needed] The **Indus Valley civilization** developed the modern **decimal** system, including the concept of **zero**.

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**Mayan numerals**

The earliest uses of mathematics were in **trading**, **land measurement**, **painting** and **weaving** patterns and the recording of **time** and nothing much more advanced until around 3000BC onwards when the **Babylonians** and **Egyptians** began using arithmetic, algebra and geometry for **taxation** and other financial calculations,
building and construction and astronomy. The systematic study of mathematics in its own right began with the Ancient Greeks between 600 and 300BC.

Mathematics has since been greatly extended, and there has been a fruitful interaction between mathematics and science, to the benefit of both. Mathematical discoveries have been made throughout history and continue to be made today. According to Mikhail B. Sevryuk, in the January 2006 issue of the Bulletin of the American Mathematical Society, "The number of papers and books included in the Mathematical Reviews database since 1940 (the first year of operation of MR) is now more than 1.9 million, and more than 75 thousand items are added to the database each year. The overwhelming majority of works in this ocean contain new mathematical theorems and their proofs."

Inspiration, pure and applied mathematics, and aesthetics

Main article: Mathematical beauty

Sir Isaac Newton (1643-1727), an inventor of infinitesimal calculus.

Mathematics arises from many different kinds of problems. At first these were found in commerce, land measurement, architecture and later astronomy; nowadays, all sciences suggest problems studied by mathematicians, and many problems arise within mathematics itself. For example, the physicist Richard Feynman invented the path integral formulation of quantum mechanics using a combination of mathematical reasoning and physical insight, and today's string theory, a still-developing scientific theory which attempts to unify the four fundamental forces of nature, continues to inspire new mathematics. Some mathematics is only relevant in the area that inspired it, and is applied to solve further problems in that area. But often mathematics inspired by one area proves useful in many areas, and joins the general stock of mathematical concepts. A distinction is often made between pure mathematics and applied mathematics. However pure mathematics topics often turn out to have applications, e.g. number theory in cryptography. This remarkable fact that even the "purest" mathematics often turns out to have practical applications is what Eugene Wigner has called "the unreasonable effectiveness of mathematics." As in most areas of study, the explosion of knowledge in the scientific age has led to specialization: there are now hundreds of specialized areas in mathematics and the latest Mathematics Subject Classification runs to 46 pages. Several areas of applied mathematics have merged with related traditions outside of mathematics and become disciplines in their own right, including statistics, operations research, and computer science.

For those who are mathematically inclined, there is often a definite aesthetic aspect to much of mathematics. Many mathematicians talk about the elegance of mathematics, its intrinsic aesthetics and inner beauty. Simplicity and generality are valued. There is beauty in a simple and elegant proof, such as Euclid's proof that there are infinitely many prime numbers, and in an elegant numerical method that speeds calculation, such as the fast Fourier transform. G. H. Hardy in A Mathematician's Apology expressed the belief that these aesthetic considerations are, in themselves, sufficient to justify the study of pure mathematics. He identified
criteria such as significance, unexpectedness, inevitability, and economy as factors that contribute to a
mathematical aesthetic.\textsuperscript{[18]} Mathematicians often strive to find proofs of theorems that are particularly
elegant, a quest Paul Erdős often referred to as finding proofs from "The Book" in which God had written
down his favorite proofs.\textsuperscript{[19][20]} The popularity of recreational mathematics is another sign of the pleasure
many find in solving mathematical questions.

**Notation, language, and rigor**

Leonhard Euler, who created and popularised much of the mathematical notation used today

*Main article: Mathematical notation*

Most of the mathematical notation in use today was not invented until the 16th century.\textsuperscript{[21]} Before that,
mathematics was written out in words, a painstaking process that limited mathematical discovery.\textsuperscript{[22]} Euler
(1707–1783) was responsible for many of the notations in use today. Modern notation makes mathematics
much easier for the professional, but beginners often find it daunting. It is extremely compressed: a few
symbols contain a great deal of information. Like musical notation, modern mathematical notation has a
strict syntax (which to a limited extent varies from author to author and from discipline to discipline) and
encodes information that would be difficult to write in any other way.

Mathematical language can also be hard for beginners. Words such as or and only have more precise
meanings than in everyday speech. Moreover, words such as open and field have been given specialized
mathematical meanings. Mathematical jargon includes technical terms such as homeomorphism and
integrable. But there is a reason for special notation and technical jargon: mathematics requires more
precision than everyday speech. Mathematicians refer to this precision of language and logic as "rigor".

The infinity symbol $\infty$ in several typefaces.
Mathematical proof is fundamentally a matter of rigor. Mathematicians want their theorems to follow from axioms by means of systematic reasoning. This is to avoid mistaken "theorems", based on fallible intuitions, of which many instances have occurred in the history of the subject. The level of rigor expected in mathematics has varied over time: the Greeks expected detailed arguments, but at the time of Isaac Newton the methods employed were less rigorous. Problems inherent in the definitions used by Newton would lead to a resurgence of careful analysis and formal proof in the 19th century. Misunderstanding the rigor is a cause for some of the common misconceptions of mathematics. Today, mathematicians continue to argue among themselves about computer-assisted proofs. Since large computations are hard to verify, such proofs may not be sufficiently rigorous.

Axioms in traditional thought were "self-evident truths", but that conception is problematic. At a formal level, an axiom is just a string of symbols, which has an intrinsic meaning only in the context of all derivable formulas of an axiomatic system. It was the goal of Hilbert's program to put all of mathematics on a firm axiomatic basis, but according to Gödel's incompleteness theorem every (sufficiently powerful) axiomatic system has undecidable formulas; and so a final axiomatization of mathematics is impossible. Nonetheless mathematics is often imagined to be (as far as its formal content) nothing but set theory in some axiomatization, in the sense that every mathematical statement or proof could be cast into formulas within set theory.

Mathematics as science

Carl Friedrich Gauss, himself known as the "prince of mathematicians", referred to mathematics as "the Queen of the Sciences".

Carl Friedrich Gauss referred to mathematics as "the Queen of the Sciences". In the original Latin Regina Scientiarum, as well as in German Königin der Wissenschaften, the word corresponding to science means (field of) knowledge. Indeed, this is also the original meaning in English, and there is no doubt that mathematics is in this sense a science. The specialization restricting the meaning to natural science is of later date. If one considers science to be strictly about the physical world, then mathematics, or at least pure mathematics, is not a science. Albert Einstein stated that "as far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality."

Many philosophers believe that mathematics is not experimentally falsifiable, and thus not a science according to the definition of Karl Popper. However, in the 1930s important work in mathematical logic showed that mathematics cannot be reduced to logic, and Karl Popper concluded that "most mathematical theories are, like those of physics and biology, hypothetico-deductive: pure mathematics therefore turns out to be much closer to the natural sciences whose hypotheses are conjectures, than it seemed even recently." Other thinkers, notably Imre Lakatos, have applied a version of falsificationism to mathematics itself.
An alternative view is that certain scientific fields (such as theoretical physics) are mathematics with axioms that are intended to correspond to reality. In fact, the theoretical physicist, J. M. Ziman, proposed that science is public knowledge and thus includes mathematics. In any case, mathematics shares much in common with many fields in the physical sciences, notably the exploration of the logical consequences of assumptions. Intuition and experimentation also play a role in the formulation of conjectures in both mathematics and the (other) sciences. Experimental mathematics continues to grow in importance within mathematics, and computation and simulation are playing an increasing role in both the sciences and mathematics, weakening the objection that mathematics does not use the scientific method. In his 2002 book A New Kind of Science, Stephen Wolfram argues that computational mathematics deserves to be explored empirically as a scientific field in its own right.

The opinions of mathematicians on this matter are varied. Many mathematicians feel that to call their area a science is to downplay the importance of its aesthetic side, and its history in the traditional seven liberal arts; others feel that to ignore its connection to the sciences is to turn a blind eye to the fact that the interface between mathematics and its applications in science and engineering has driven much development in mathematics. One way this difference of viewpoint plays out is in the philosophical debate as to whether mathematics is created (as in art) or discovered (as in science). It is common to see universities divided into sections that include a division of Science and Mathematics, indicating that the fields are seen as being allied but that they do not coincide. In practice, mathematicians are typically grouped with scientists at the gross level but separated at finer levels. This is one of many issues considered in the philosophy of mathematics.

Mathematical awards are generally kept separate from their equivalents in science. The most prestigious award in mathematics is the Fields Medal, established in 1936 and now awarded every 4 years. It is often considered the equivalent of science's Nobel Prizes. The Wolf Prize in Mathematics, instituted in 1978, recognizes lifetime achievement, and another major international award, the Abel Prize, was introduced in 2003. These are awarded for a particular body of work, which may be innovation, or resolution of an outstanding problem in an established field. A famous list of 23 such open problems, called "Hilbert's problems", was compiled in 1900 by German mathematician David Hilbert. This list achieved great celebrity among mathematicians, and at least nine of the problems have now been solved. A new list of seven important problems, titled the "Millennium Prize Problems", was published in 2000. Solution of each of these problems carries a $1 million reward, and only one (the Riemann hypothesis) is duplicated in Hilbert's problems.

Fields of mathematics

An abacus, a simple calculating tool used since ancient times.

Mathematics can, broadly speaking, be subdivided into the study of quantity, structure, space, and change (i.e. arithmetic, algebra, geometry, and analysis). In addition to these main concerns, there are also subdivisions dedicated to exploring links from the heart of mathematics to other fields: to logic, to set theory (foundations), to the empirical mathematics of the various sciences (applied mathematics), and more recently to the rigorous study of uncertainty.
Quantity

The study of quantity starts with numbers, first the familiar natural numbers and integers ("whole numbers") and arithmetical operations on them, which are characterized in arithmetic. The deeper properties of integers are studied in number theory, from which come such popular results as Fermat's Last Theorem. Number theory also holds two problems widely considered to be unsolved: the twin prime conjecture and Goldbach's conjecture.

As the number system is further developed, the integers are recognized as a subset of the rational numbers ("fractions"). These, in turn, are contained within the real numbers, which are used to represent continuous quantities. Real numbers are generalized to complex numbers. These are the first steps of a hierarchy of numbers that goes on to include quaternions and octonions. Consideration of the natural numbers also leads to the transfinite numbers, which formalize the concept of "infinity". Another area of study is size, which leads to the cardinal numbers and then to another conception of infinity: the aleph numbers, which allow meaningful comparison of the size of infinitely large sets.

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Structure

Many mathematical objects, such as sets of numbers and functions, exhibit internal structure. The structural properties of these objects are investigated in the study of groups, rings, fields and other abstract systems, which are themselves such objects. This is the field of abstract algebra. An important concept here is that of vectors, generalized to vector spaces, and studied in linear algebra. The study of vectors combines three of the fundamental areas of mathematics: quantity, structure, and space. A number of ancient problems concerning Compass and straightedge constructions were finally solved using Galois theory.

Space

The study of space originates with geometry – in particular, Euclidean geometry. Trigonometry is the branch of mathematics that deals with relationships between the sides and the angles of triangles and with the trigonometric functions; it combines space and numbers, and encompasses the well-known Pythagorean theorem. The modern study of space generalizes these ideas to include higher-dimensional geometry, non-Euclidean geometries (which play a central role in general relativity) and topology. Quantity and space both play a role in analytic geometry, differential geometry, and algebraic geometry. Within differential geometry are the concepts of fiber bundles and calculus on manifolds, in particular, vector and tensor calculus. Within algebraic geometry is the description of geometric objects as solution sets of polynomial equations,
combining the concepts of quantity and space, and also the study of topological groups, which combine structure and space. Lie groups are used to study space, structure, and change. Topology in all its many ramifications may have been the greatest growth area in 20th century mathematics; it includes point-set topology, set-theoretic topology, algebraic topology and differential topology. In particular, instances of modern day topology are metrizability theory, axiomatic set theory, homotopy theory, and Morse theory. Topology also includes the now solved Poincaré conjecture and the controversial four color theorem, whose only proof, by computer, has never been verified by a human.

Change

Understanding and describing change is a common theme in the natural sciences, and calculus was developed as a powerful tool to investigate it. Functions arise here, as a central concept describing a changing quantity. The rigorous study of real numbers and functions of a real variable is known as real analysis, with complex analysis the equivalent field for the complex numbers. Functional analysis focuses attention on (typically infinite-dimensional) spaces of functions. One of many applications of functional analysis is quantum mechanics. Many problems lead naturally to relationships between a quantity and its rate of change, and these are studied as differential equations. Many phenomena in nature can be described by dynamical systems; chaos theory makes precise the ways in which many of these systems exhibit unpredictable yet still deterministic behavior.

Foundations and philosophy

In order to clarify the foundations of mathematics, the fields of mathematical logic and set theory were developed. Mathematical logic includes the mathematical study of logic and the applications of formal logic to other areas of mathematics; set theory is the branch of mathematics that studies sets or collections of objects. Category theory, which deals in an abstract way with mathematical structures and relationships
between them, is still in development. The phrase "crisis of foundations" describes the search for a rigorous foundation for mathematics that took place from approximately 1900 to 1930.[33] Some disagreement about the foundations of mathematics continues to present day. The crisis of foundations was stimulated by a number of controversies at the time, including the controversy over Cantor's set theory and the Brouwer-Hilbert controversy.

Mathematical logic is concerned with setting mathematics on a rigorous axiomatic framework, and studying the results of such a framework. As such, it is home to Gödel's second incompleteness theorem, perhaps the most widely celebrated result in logic, which (informally) implies that any formal system that contains basic arithmetic, if sound (meaning that all theorems that can be proven are true), is necessarily incomplete (meaning that there are true theorems which cannot be proved in that system). Gödel showed how to construct, whatever the given collection of number-theoretical axioms, a formal statement in the logic that is a true number-theoretical fact, but which does not follow from those axioms. Therefore no formal system is a true axiomatization of full number theory.[citation needed] Modern logic is divided into recursion theory, model theory, and proof theory, and is closely linked to theoretical computer science.

Discrete mathematics

Discrete mathematics is the common name for the fields of mathematics most generally useful in theoretical computer science. This includes, on the computer science side, computability theory, computational complexity theory, and information theory. Computability theory examines the limitations of various theoretical models of the computer, including the most powerful known model – the Turing machine. Complexity theory is the study of tractability by computer; some problems, although theoretically solvable by computer, are so expensive in terms of time or space that solving them is likely to remain practically unfeasible, even with rapid advance of computer hardware. Finally, information theory is concerned with the amount of data that can be stored on a given medium, and hence deals with concepts such as compression and entropy.

On the purely mathematical side, this field includes combinatorics and graph theory.

As a relatively new field, discrete mathematics has a number of fundamental open problems. The most famous of these is the "P=NP?" problem, one of the Millennium Prize Problems.[34]
Applied mathematics

**Applied mathematics** considers the use of abstract mathematical tools in solving concrete problems in the sciences, business, and other areas.

Applied mathematics has significant overlap with the discipline of **statistics**, whose theory is formulated mathematically, especially with probability theory. Statisticians (working as part of a research project) "create data that makes sense" with random sampling and with randomized experiments; the design of a statistical sample or experiment specifies the analysis of the data (before the data be available). When reconsidering data from experiments and samples or when analyzing data from observational studies, statisticians "make sense of the data" using the art of modelling and the theory of inference – with model selection and estimation; the estimated models and consequential predictions should be tested on new data.\(^\text{[3]}\)

**Computational mathematics** proposes and studies methods for solving mathematical problems that are typically too large for human numerical capacity. **Numerical analysis** studies methods for problems in analysis using ideas of functional analysis and techniques of approximation theory; numerical analysis includes the study of approximation and discretization broadly with special concern for rounding errors. Other areas of computational mathematics include computer algebra and symbolic computation.