Grade 12 Pre-Calculus Mathematics (40S)

A Course for Independent Study
Manitoba Education, Citizenship and Youth Cataloguing in Publication Data

510 Grade 12 pre-calculus mathematics (40S) : a course for independent study

Previously published as : Senior 4 pre-calculus mathematics (40S) : a course for distance learning.


Copyright © 2007, the Crown in Right of Manitoba as represented by the Minister of Education, Citizenship and Youth.

Manitoba Education, Citizenship and Youth
School Programs Division
1970 Ness Avenue
Winnipeg, Manitoba R3J 0Y9.

Every effort has been made to acknowledge original sources and to comply with copyright law. If cases are identified where this has not been done, please notify Manitoba Education, Citizenship and Youth. Errors or omissions will be corrected in a future edition. Sincere thanks to the authors and publishers who allowed their original material to be adapted or reproduced.
Acknowledgements

Manitoba Education, Citizenship and Youth gratefully acknowledges the contributions of the following individuals in the development of Grade 12 Pre-Calculus Mathematics: A Course for Independent Study (40S).

Writer
William Korytowski Consultant Winnipeg, Manitoba

Content Editor
Harry Dmytryshyn Consultant Winnipeg S.D. No. 1
(Field Validation Version)
Carolyn Wilkinson Consultant Winnipeg, Manitoba
(Final Version)

Members of the Development Team
John Barsby St. John’s-Ravenscourt School Independent Schools
Dave Klassen R.D. Parker Collegiate Mystery Lake S.D. No. 2355
Viviane Leonard Collège Beliveau St. Boniface S.D. No. 4
Jayesh Manier Oak Park High School Assiniboine South S.D. No. 3
Don Nichol Whitewater School Agassiz S.D. No. 13
Hilliard Sawchuk Consultant Winnipeg, Manitoba
Don Trim University of Manitoba University of Manitoba
Alan Wells Consultant Winnipeg, Manitoba
Abdou Daoudi Bureau de l’éducation française Manitoba Education, Training and Youth
Katharine Tetlock School Programs Division Manitoba Education, Training and Youth
Wayne Watt School Programs Division Manitoba Education, Training and Youth

Manitoba Education, Training and Youth Staff
School Programs Division
Carole Bilyk Project Leader Curriculum Unit
(Final Version) Program Development Branch
Lee-Ila Bothe Coordinator Production Support Unit
Paul Cuthbert Project Manager Distance Learning and
(Field Validation Version) Information Technologies Unit
Lynn Harrison Desktop Publisher Program Development Branch
Gilles Landry Project Manager Distance Learning and
Project Leader: Information Technologies Unit
Katharine Tetlock Program Development Branch
(Final Version)
Contents

Acknowledgements iii

Introduction 3
Formula Sheet 9
Module Test Cover Sheets

Module 1: Transformations
Introduction 3
Lesson 1: Translations 5
Lesson 2: Reflections and Symmetry 13
Lesson 3: Absolute Value and Piece-Wise Functions 29
Lesson 4: Stretches and Compressions 39
Lesson 5: Reciprocals 51
Lesson 6: Combinations of Transformations 61
Review 73

Module 1 Answer Key

Module 2: Circular Functions I
Introduction 3
Lesson 1: Trigonometric Values of Special Angles 5
Lesson 2: The Unit Circle 13
Lesson 3: The Circular Functions 21
Lesson 4: Radian Measures 35
Lesson 5: Sine and Cosine Graphs 47
Lesson 6: Graphs of the Remaining Four Circular Functions 61
Lesson 7: The Inverse Circular Functions 65
Review 81

Module 2 Answer Key

Module 3: Circular Functions II
Introduction 3
Lesson 1: Elementary Identities 5
Lesson 2: Using Elementary Identities 15
Lesson 3: Sum and Difference Identities 21
Lesson 4: Double Angle Identities 29
Lesson 5: Modelling with Trigonometric Functions 35
Review 43

Module 3 Answer Key
Module 4: Exponential and Logarithmic Functions I

Introduction 3
Lesson 1: Graphing Exponential Functions 5
Lesson 2: A Special Exponential Function 13
Lesson 3: The Logarithmic Function 19
Lesson 4: An Algebraic Approach to Logarithms 25
Lesson 5: The Logarithmic Theorems 31
Lesson 6: Finding Logarithms Using a Calculator 39
Lesson 7: Solving Exponential and Logarithmic Equations 45
Review 51

Module 4 Answer Key

Module 5: Exponential and Logarithmic Functions II

Introduction 3
Lesson 1: Applications of Exponential Functions 5
Lesson 2: More Applications of Exponential Functions 15
Lesson 3: Geometric Sequences 25
Lesson 4: The Sum of a Geometric Series 33
Lesson 5: The Sum of an Infinite Geometric Series 41
Review 49

Module 5 Answer Key

Module 6: Permutations and Combinations

Introduction 3
Lesson 1: Fundamental Counting Principle 5
Lesson 2: The Formula \( ^nP_r \) and Factorial Notation 15
Lesson 3: Grouped Permutations and Circular Permutations 25
Lesson 4: Like Objects 35
Lesson 5: Combinations 41
Lesson 6: Binomial Theorem 51
Review 61

Module 6 Answer Key
Module 7: Probability
  Introduction 3
  Lesson 1: Review of Probability Ideas 5
  Lesson 2: The Probability Laws 13
  Lesson 3: Two Special Cases 21
  Lesson 4: Conditional Probability 25
  Lesson 5: Practice Problems of Compound Events 31
  Lesson 6: Probabilities Using Permutations and Combinations 37
  Review 41

Module 7 Answer Key

Module 8: Conic Sections
  Introduction 3
  Lesson 1: Review of Circles 5
  Lesson 2: The Ellipse and Hyperbola 11
  Lesson 3: Parabolas and Classifying the Conic Sections 23
  Lesson 4: Converting from General to Standard Form 29
  Review 37

Module 8 Answer Key

Module Tests
  Module 1 Hand-in
  Module 2
  Module 3 Hand-in
  Module 4
  Module 5 Hand-in
  Module 6
  Module 7 Hand-in
  Module 8

Module Tests Answer Key
  Module 2
  Module 4
  Module 6
  Module 8
<table>
<thead>
<tr>
<th>viii</th>
<th>Contents</th>
<th>Grade 12 Pre-Calculus Mathematics</th>
</tr>
</thead>
</table>

Notes
GRADE 12 PRE-CALCULUS
MATHEMATICS (40S)

Introduction
Welcome to the *Grade 12 Pre-Calculus Mathematics* course for distance learning offered through the School Programs Division, Manitoba Education, Citizenship and Youth. **It is expected that you have successfully completed *Grade 11 Pre-Calculus Mathematics*.**

As a student in a course for distance learning, you have taken on a dual role—that of a student and a teacher. As a student, you are responsible for mastering the lessons and completing the exercises assigned at the end of each lesson. As a teacher, you are responsible for checking your work carefully and noting the nature of your errors. Finally, you must work diligently to overcome your difficulties.

You should seek out a study partner for this course. Most students find that a study partner helps them get through the course with greater success. This study partner can help you correct your assignments and module self-tests, as well as help you prepare for the examinations.

*Grade 12 Pre-Calculus Mathematics* is one of four possible Grade 12 mathematics courses in Manitoba (the other three courses are *Grade 12 Applied Mathematics, Grade 12 Consumer Mathematics*, and *Grade 12 Accounting Systems*). This course contains many new and interesting topics in mathematics. You will need to use many of the skills and procedures that you have already learned to solve some of the problems you will find in the exercises. On completion of this course, successful students will be well prepared to study post-secondary mathematics.

This course is divided into eight modules. Each module contains lessons that are followed by assignments. **It is recommended that you complete all of the assigned exercises.** Answer keys are provided to the exercises and are found at the end of each module.
The eight modules are as follows:

Module 1: Transformations
Module 2: Circular Functions I
Module 3: Circular Functions II
Module 4: Exponential and Logarithmic Functions I
Module 5: Exponential and Logarithmic Functions II
Module 6: Permutations and Combinations
Module 7: Probability
Module 8: Conic Sections

The table of contents outlines the topics found in this course. Every student enrolled in Grade 12 Pre-Calculus Mathematics is required to complete all eight modules. Each module ends with a test.

These tests should be written without the aid of any books. Your performance on these tests will give you an indication of how well you understand the material. Your study partner can help you by marking some of these tests. Answer keys for Module Tests 2, 4, 6, and 8 are provided in the Module Tests Answer Keys section at the end of the course. You should then correct all errors and use these tests to help you prepare for the midterm and final examinations. Note that Module Tests 1, 3, 5, and 7 are to be sent to the Tutor/Marker as soon as each one is completed. Therefore, no answer keys are provided for these tests.

**Tutor/Marker**

The person who marks your tests and exams is your tutor/marker. This person is also available to help you with your learning. When you register for the course, you will receive a letter that gives you the name and contact information for the tutor/marker for Grade 12 Pre-Calculus Mathematics. Please take advantage of all the resources provided by the Independent Study Office.
Calculator Use
You will need a scientific calculator for this course. A graphing calculator may be helpful but it is not necessary. References made to the graphing calculator in the course are in optional sections. Many of the exercises ask for exact answers where calculator use is not required.

You are permitted to use a scientific calculator for all tests and exams. You are not asked questions requiring a graphing calculator.

Formula Sheet
A formula sheet is provided for the exams. You may also use it for your tests. A copy of the formula sheet is included at the end of this introductory section.

Evaluation
Your final mark in this course will be based on the results of four hand-in tests, and two examinations: a midterm and a final.

The value of these tests and examinations are shown below.

First Term:  Hand-in Tests 1 and 3, 10% each  20%
            Midterm Examination after Module 4  20%
                  (Based on Modules 1 through 4)
Second Term: Hand-in Tests 5 and 7, 10% each  20%
            Final Examination after Module 8  40%
                  (Based on Modules 1 through 8)

Total 100%

The module tests after Modules 1, 3, 5, and 7 must be completed, and then sent in to your Tutor/Marker. You should also note that the final examination is cumulative, meaning that it is based on the entire course.

You are required to send a coloured cover sheet with each hand-in test. Cover sheets can be found after page 6 of the Introduction.
Guide Graphics

Graphics have been placed inside the margins of the course to identify a specific task. Each graphic has a specific purpose to help guide you.

The significance of each guide graphic is described below.

- **Assignment:** You are required to do the assignment questions that accompany this graphic.

- **Note:** This graphic will appear when there is a direction or explanation that you should note carefully.

- **Test Time:** This graphic alerts students that it is time to write a test or prepare for a test.

- **Send In:** This graphic indicates that you must send in the assignment or self-test for correcting.

- **Study/Review:** This graphic is to remind you that you should review your material for a test or examination.

- **Check:** Check your answers against Answer Key provided for this lesson.

- **Exam Time:** When this graphic appears, it is time to write an examination

Cautionary Note

Some of the activities in this course involve chance and probability. In some families and communities, the connection between probability and gambling may be problematic; for example, parents/guardians may not approve of playing cards, dice, or prize money. As an alternative, students can use numbered index cards, number cubes, or points or credits.
Applying for Exams

- If you are attending school, ask your Independent Study Option (ISO) facilitator to add your name to the ISO examination eligibility list. Do this at least three weeks prior to the next scheduled examination week.

- If you are not attending school, check the Examination Request Form for options available to you. Fill in this form and mail or fax three weeks before you are ready to write the Grade 12 Pre-Calculus Mathematics Midterm or Final Examination. The address is:
  
  ISO Tutor/Marker
  555 Main Street
  WINKLER MB R6W 1C4
  Fax: 204-325-1719

Contact Information

Use the following mailing address for any materials that you are forwarding to your tutor/marker:

Distance Learning and Information Technologies Unit
555 Main Street
WINKLER MB R6W 1C4
Telephone: (204) 325-1700
Toll-free: (800) 465-9915
Fax: (204) 325-1719

Website: <www.edu.gov.mb.ca/k12/dl/iso/index.html>

This website contains information concerning policies and procedures for the Independent Study Option as well as other useful information. It also includes a forms section where you can download the application form for writing exams.


**Formula Sheet**

**Senior 4 Pre-Calculus Mathematics (40S)**

\[ s = \theta r \]

\[
\sin^2 \theta + \cos^2 \theta = 1
\]

\[ \tan^2 \theta + 1 = \sec^2 \theta \]

\[ 1 + \cot^2 \theta = \csc^2 \theta \]

\[
\sin (\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta
\]

\[
\cos (\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta
\]

\[
\tan (\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}
\]

\[
\sin (\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta
\]

\[
\cos (\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta
\]

\[
\tan (\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}
\]

\[ \sin 2\alpha = 2 \sin \alpha \cos \alpha \]

\[ \cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha \]

\[ \cos 2\alpha = 1 - 2 \sin^2 \alpha \]

\[ \cos 2\alpha = 2 \cos^2 \alpha - 1 \]

\[ \tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha} \]

\[ P(n, r) = \frac{n!}{(n-r)!} \]

\[ C(n, r) = \frac{n!}{r!(n-r)!} \]

\[ t_{k+1} = \sum_{k=0}^{n} a^n b^k \]

\[ P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) \]

\[ P(A \text{ and } B) = P(A) \cdot P(B|A) \]

\[ A = P \left(1 + \frac{r}{n}\right)^n \]

\[ A = Pe^{rt} \]

\[ e = 2.718 \text{ 28} \]

\[ \log_a (MN) = \log_a M + \log_a N \]

\[ \log_a \left(\frac{M}{N}\right) = \log_a M - \log_a N \]

\[ \log_a M = \frac{\log_b M}{\log_b a} \]

\[ (x-h)^2 + (y-k)^2 = r^2 \]

\[ \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1, \ a > b \]

\[ (x-h)^2 + \frac{(y-k)^2}{a^2} = 1, \ a > b \]

\[ \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \]

\[ (y-k)^2 - \frac{(x-h)^2}{b^2} = 1 \]

\[ (x-h) = a(x-h)^2 \]

\[ (x-h) = a(y-k)^2 \]

\[ t_n = t_1 r^{n-1} \]

\[ S_n = \frac{t_1 (1-r^n)}{1-r} = \frac{t_1 (r^n - 1)}{r - 1} \]

\[ S_n = \frac{t_1 - t_2 r}{1-r} \]

\[ S_n = \frac{t_1}{1-r}, \ |r| < 1 \]
Lesson 1

Translations

Outcomes
Upon completing this lesson, you will be able to
• describe how a translation affects the graph and properties of a function
• sketch the translation of a function
• state the translation that produced a new sketch from the given sketch

Overview
You have encountered translations when you sketched the following three functions in Grade 11 Pre-Calculus Mathematics.

![Graphs of functions](https://via.placeholder.com/150)

- $y = x^2$
- $y = x^2 - 1$
- $y = (x - 1)^2$
Defining Translations

In \( y = x^2 - 1 \), each \( y \)-value is 1 less than the corresponding \( y \)-value in \( y = x^2 \). Therefore the graph of \( y = x^2 - 1 \) is 1 unit lower than the graph of \( y = x^2 \). We say that the graph of \( y = x^2 \) has been **translated** 1 unit down. The function \( y = x^2 - 1 \) is a **vertical translation** of the function, \( y = x^2 \).

Similarly, \( y = (x - 1)^2 \) is a **horizontal translation** of \( y = x^2 \). The graph of \( y = x^2 \) has been shifted one unit to the right.

**Definition:** A **translation** is a transformation of a geometric figure in which every point is moved the same distance in the same direction.

**Example 1**

Use the sketch of \( y = x^2 \) to sketch

a) \( y = x^2 + 2 \)

b) \( y = (x + 2)^2 \)

**Note:** A graphing calculator is not required for any part of this course.

**Solution**

a) 

As evident from the graphs, the sketch of \( y = x^2 + 2 \) is a vertical translation of two units up, and the graph of \( y = (x + 2)^2 \) is a horizontal translation of two units to the left of the graph of \( y = x^2 \).

Notice that with the vertical translation the “2” is outside the brackets, and affects the \( y \)-values by an amount of 2. With the horizontal translation, the “2” is inside the brackets and affects the \( x \)-values, not by an amount of 2 but rather by –2.
Vertical translations are often easier and more natural to understand. Is there some way to remember in which direction to shift a horizontal translation? One suggestion is to follow the vertex or the intercepts, whichever is more convenient. In the above examples, the smallest value of a squared quantity is zero, thus producing the vertex. Let’s follow the vertex as shown in the chart below.

<table>
<thead>
<tr>
<th>Quadratic Equation</th>
<th>The Value of x Which Makes y = 0</th>
<th>Effects on Graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = x^2 )</td>
<td>0</td>
<td>basic graph</td>
</tr>
<tr>
<td>( y = (x + 2)^2 )</td>
<td>-2</td>
<td>2 units left</td>
</tr>
<tr>
<td>( y = (x - 2)^2 )</td>
<td>2</td>
<td>2 units right</td>
</tr>
</tbody>
</table>

**Example 2**

Use the given sketch of \( f(x) = x^3 \) to sketch the following functions. In this example, there is no vertex. Try following the \( x \)-intercept instead. Remember that the \( x \)-intercepts are also called the zeros of the function.

\[ f(x) = x^3 \]

---

a) \( g(x) = x^3 - 2 \)

b) \( h(x) = (x - 1)^3 \)

c) \( k(x) = (x + 3)^3 \)
Solution

a) \[ g(x) = x^3 - 2 \]

b) \[ h(x) = (x - 1)^3 \]

c) \[ k(x) = (x + 3)^3 \]

Example 3

Use the given sketch of \( f(x) \) to sketch the following functions. Again, try translating the \( x \)- and \( y \)-intercepts as a tool for making the graphing easier.

a) \( g(x) = f(x + 3) \)

b) \( m(x) = f(x) + 2 \)

c) \( n(x) = f(x - 5) \)
Solution

a) 

\[ g(x) \]

\[ \begin{array}{c}
\text{Translation} \\
\text{Effect on Graph} \\
\text{Effect on } (x, y) \\
f(x) + k \\
\text{Vertical translation of } k \text{ units:} \\
\cdot \text{up if } k > 0 \\
\cdot \text{down if } k < 0 \\
(x, y + k) \\
f(x - h) \\
\text{Horizontal translation of } h \text{ units:} \\
\cdot \text{to the left if } h < 0 \\
\cdot \text{to the right if } h > 0 \\
(x + h, y)
\end{array} \]
Assignment

1. Given the sketch of \( f(x) \) drawn below, sketch each of the following functions.
   a) \( f(x - 5) \)
   b) \( f(x) - 5 \)
   c) \( f(x) + 5 \)
   d) \( f(x + 5) \)
   e) \( f(x - 5) - 5 \)
   f) \( f(x + 5) + 5 \)

2. Let \( f(x) = x^2 + 2 \). Sketch each of the following functions.
   a) \( f(x) \)
   b) \( f(x) - 6 \)
   c) \( f(x + 1) \)
   d) \( f(x - 2) - 3 \)

3. For each of the functions in Question 2 state the properties of the function: the domain, the range, and the values of the intercepts. (Recall: To find the \( y \)-intercept, let \( x = 0 \) and solve for \( y \). To find the \( x \)-intercept, let \( y = 0 \) and solve for \( x \). For more practice, see Lesson 3.)

4. Each of graphs (a), (b), and (c) represents a translation of the given function, \( g(x) \), drawn below. Write an expression for each new function in terms of \( g(x) \).
5. If \( f(x) = x^3 + 3x^2 - x + 6 \), write an unsimplified equation for
   a) \( g(x) \) which has the same graph as \( f(x) \) moved two units to the left.
   b) \( h(x) \) which has the same graph as \( f(x) \) moved three units down.
   c) \( m(x) \) which has the same graph as \( f(x) \) moved two units to the right and one unit up.

6. How are the graphs of \( g(x) = \frac{|x|}{x} \) and \( n(x) = \frac{|x+3|}{x+3} \) related?

   (For more practice, see Lesson 3, Assignment, questions 3, 4, and 5.)
7. Below is the graph of \( f(x) = x^3 - x \). Sketch the graph of \( g(x) = (x + 2)^3 - (x + 2) \).

8. Is the translation of a function still a function?

Check your answers in the Module 1 Answer Key.