1 Overview

In this section, we will introduce some basic mathematical notation that will allow us to express arguments mathematically. We will begin translation and evaluation of arguments into truth tables. Having used truth tables to motivate an explanation of the criteria for a valid argument, we will develop a method for doing proofs.

2 Deductive reasoning and logical connectives

Consider the following example.

Example 1 It will either rain or snow tomorrow. It is too warm for snow. Therefore, it will rain.

We arrive at a conclusion based on the assumption that the premises are correct. Consider Example 2. There are 2 premises: “It will either rain or snow tomorrow.” and “It is too warm for snow.” Is the conclusion correct? Suppose it neither rains nor snows tomorrow. The conclusion in this case would be false; however, so would the first premise. So the conclusion only holds when the premises are true.

We say that an argument is valid if when the premises are true the conclusion must also be true. Is the following example valid?

Example 2 Either Miss Scarlet is guilty or Mr. Green is guilty. Either Mr. Green is guilty or Mrs. Peacock is guilty. Therefore, either Miss Scarlet or Mrs. Peacock is guilty.

Let’s assume the premises are true. Must the conclusion be true? No. What if Mr. Green is guilty? That is possible under the premises but would not lead to this conclusion. So an argument is invalid if we can show that it is possible that when the premises are true, the conclusion can be false.

So far, we’ve been writing everything out using words. This can get a little confusing because of the large number of words. Often, it is convenient to use symbols to represent concepts. Reconsider
Example ??: Suppose we let $P$ represent the possibility of rain and $Q$ represent the possibility of snow. Then this argument has the form:

\[
\begin{align*}
P & \text{ or } Q \\
\text{not } Q \\
\text{Therefore, } P.
\end{align*}
\]

Just as we can use letters like $P$ and $Q$ to represent concepts, we can also introduce some other symbolic notation. Therefore, $P \lor Q$ means "$P$ or $Q$" and the negation of $P$ is $\neg P$. Note that just

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
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<tr>
<td>$\lor$</td>
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<td>$\land$</td>
<td>and</td>
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<td>$\neg$</td>
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as with math, placement of commas is important.

**Exercise 3** Analyze the logical form of the following sentence: Either Bill is at work and Jane isn’t or Jane is at work and Bill isn’t.

### 3 Truth Tables

Recall that an argument is valid if the premises cannot all be true without the conclusion also being true. Truth tables can be very helpful in determining whether an argument is valid. First we’ll show how to construct truth tables. Then we’ll show how to use them to evaluate arguments.

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<tr>
<th>$P$</th>
<th>$Q$</th>
<th>$P \land Q$</th>
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In this table, we are looking at all the possible combinations of $P$ and $Q; each can either be true or false. We see that if $P$ is false (ie, doesn’t occur) and $Q$ is false, then $P \land Q$ must also be false. Similarly, if just one of the elements is false, $P \land Q$ must be false. Only if both are true can $P \land Q$ be true. This isn’t too complicated. Neither is the truth table for $\neg P$, nor for $P \lor Q$, 2
Exercise 4 Make the truth table for \( \neg(P \lor \neg Q) \).

Exercise 5 Make the truth table for \( \neg(P \land \neg Q) \lor \neg R \).

Think back to Example ?? from earlier. Make a truth table and use it to analyze whether the
argument is valid. Recall that we decided that the form of this argument was:

\[
\begin{align*}
P \lor Q \\
\neg Q \\
\therefore P
\end{align*}
\]

We said that the requirement for an argument to be valid is that if the premises are all true, the conclusion must also be true. The premises here are \((P \lor Q)\) and \(\neg Q\). The conclusion is \(P\).

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Observe that when the premises are both true, the conclusion is also true. Therefore, we have shown symbolically that this argument is valid.

**Exercise 6** Use a truth table to determine whether the following argument is valid. "Either John isn’t stupid and he is lazy, or he’s stupid. John is stupid. Therefore, John isn’t lazy.

### 3.1 Equivalencies

Many times, logical statements can be simplified, making the truth tables easier and faster to calculate. This is because symbolic logic follows many of the same properties as typical math.

- DeMorgan’s Law: \(\neg (P \land Q)\) is equivalent to \(\neg P \lor \neg Q\)
- Commutative laws: \(P \land Q\) is equivalent to \(Q \land P\)
- Associative laws: \(P \land (Q \land R)\) is equivalent to \((P \land Q) \land R\)
- Idempotent laws: \(P \land P\) is equivalent to \(P\)
• Distributive laws: \( P \land (Q \lor R) \) is equivalent to \( (P \land Q) \lor (P \land R) \)

• Double negation law: \( \neg \neg P \) is equivalent to \( P \)

You can (and should!) show that the above are equivalent.

**Exercise 7** Find simpler formulas equivalent to these formulas

\( \neg (P \lor \neg Q) \)

\( \neg (Q \land \neg P) \lor P \)

Formulas that are always true, such as \( P \lor P \), are called *tautologies*. Formulas that are always false are called *contradictions*.

### 4 Variables and Sets

So far, we've symbolized statements using simple letters. We have not accounted for the fact that a statement might include variables. Consider the statement, ”\( x \) is a prime number”. Let \( x \) be a variable. We can write this statement as \( P(x) \). By writing this in this way, rather than just \( P \), we are accounting for the fact that \( x \) can take on different values. If a statement includes more than one variable, all variables should be shown: eg, \( D(x, y) \).

**Exercise 8** Write out the following in symbolic notation: \( x \) is a man and \( y \) is a woman and \( x \) likes \( y \) but \( y \) does not like \( x \).

With dealing with statements without variables, a statement is always either true or false. This is not the case when a statement includes variables; the truth or falsity of the statement depends on the value that the variable takes. For example, \( P(7) \) is true while \( P(9) \) is false. We deal with this through the introduction of *truth sets*. First, we introduce some basic set theory.

A set is a collection of items. The objects in the set are called *elements* of the set.

**Example 9** \( A = \{3, 7, 14\} \)

\( \in \) means ”is an element of”. Therefore, in the above example, \( 7 \in A \) but \( 11 \notin A \). The order of elements in a set doesn’t matter.

**Example 10** If \( B = \{14, 7, 3\} \) then \( A = B \).

It can be tiresome to write out all the numbers in a set. We have notation that simplifies this. \( C = \{x \mid x \text{ is a prime number}\} \) is read as ”\( C=\text{the set of all } x \text{ such that } x \text{ is a prime number}”\). So, \( C \) are the values of \( x \) that make the statement ”\( x \text{ is a prime number}” \text{ true}. So the statement ”\( x \text{ is a prime number}” \text{ is an elementhood test}.”
Definition 11 \( N = \{ x \mid x \text{ is a natural number} \} \). Natural numbers the numbers that we use for counting: ie., \( N = \{ 0, 1, 2, 3, \ldots \} \)

Definition 12 \( Z = \{ x \mid x \text{ is an integer} \} \). Integers are the natural numbers, their negatives, and 0. ie., \( Z = \{ \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \} \)

Definition 13 \( Q = \{ x \mid x \text{ is a rational number} \} \). A rational number can be written as a fraction, \( \frac{p}{q} \), where \( p \) and \( q \) are integers.

Definition 14 \( R = \{ x \mid x \text{ is a real number} \} \). A real number is any number on the number line. So \( R \) consists of integers, the rational numbers, and all of the other numbers on the number line that don’t meet the above definitions, such as \( \sqrt{2} \).

You can use subscript + or − to indicate positive or negative numbers (eg.,

\[ R^+ = \{ x \mid x \text{ is a positive real number} \} \].

Go back to our example with \( C \). ”\( x \) is a prime number” is an elementhood test for the set. Any value of \( x \) that makes the statement true passes the test and thus is an element of the set. If it fails the test, it isn’t an element of the set.

Example 15 \( \{ x \mid x^2 < 9 \} \). Since \( 5^2 \not< 9 \) then \( 5 \notin \{ x \mid x^2 < 9 \} \).

Exercise 16 Translate

\[ a + b \notin \{ x \mid x \text{ is an even number} \} \]

Here is some more useful set notation.

Definition 17 \( U \) (universe) is the set of all possible values that variables can take in a particular problem.

Definition 18 \( \emptyset \) is the empty set or null set, meaning a set of no elements. \( \emptyset = \{ \} \)

Example 19 \( \{ x \in Z \mid x \neq x \} = \emptyset \)

Just as with symbolic logic, we have operations on sets.

Definition 20 Intersection: \( A \cap B = \{ x \mid x \in A \text{ and } x \in B \} \)

Definition 21 Union: \( A \cup B = \{ x \mid x \in A \text{ or } x \in B \} \)
**Definition 22** Difference: \( A \setminus B = \{ x | x \in A \text{ and } x \notin B \} \)

**Exercise 23** \( A = \{1, 2, 3, 4, 5\} \) and \( B = \{2, 4, 6, 8, 10\} \).

\[
\begin{align*}
A \cap B &= \quad \\
A \cup B &= \quad \\
A \setminus B &= \quad \\
(A \cup B) \setminus (A \cap B) &= \quad \\
(A \setminus B) \cup (B \setminus A) &= \quad \\
B \setminus A &= \\
\end{align*}
\]

**Example 24** \( A = \{ x | x \text{ is a man} \} \). \( B = \{ x | x \text{ has brown hair} \} \).

\[
\begin{align*}
A \cap B &= \quad \\
A \cup B &= \quad \\
A \setminus B &= \quad \\
\end{align*}
\]

We can also depict these concepts using Venn diagrams.

## 5 Conditional and Biconditional Connectives

\( P \rightarrow Q \) is read as ”if \( P \) then \( Q \)”. This is called a conditional statement. \( P \) is the antecedent and \( Q \) is the consequent. Consider the following example:

**Example 25** If today is Sunday, then I don’t have to go to work. Today is Sunday. Therefore, I don’t have to work today.
Putting these statements into logical form, we get:

\[
\begin{align*}
P & \rightarrow Q \\
\therefore & \quad Q
\end{align*}
\]

Exercise 26  *Put in logical form. "If it’s raining and I don’t have my umbrella, then I’ll get wet."*

You won’t be surprised to find that we can also express conditional statements through truth tables. What we see in the table is that a statement is false if the antecedent is true and the consequent is false; otherwise the statement is true. Reconsider the Sunday example.

\[
\begin{array}{ccc}
P & Q & P \rightarrow Q \\
F & F & T \\
F & T & T \\
T & F & T \\
T & T & T \\
\end{array}
\]

- Today is Sunday (T). I don’t have to go to work (T). Therefore, \(P \rightarrow Q\) is true.
- Today is Sunday (T). I have to go to work (F). Therefore, \(P \rightarrow Q\) is false.
- Today isn’t Sunday (F) but I have to go to work (F). There is nothing untrue about the statement \(P \rightarrow Q\).
- Today isn’t Sunday (F) and I don’t have to go to work (T). There is nothing untrue about the statement \(P \rightarrow Q\).

Exercise 27  *Now do the truth table for \(\neg P \lor Q\) and see what you notice.*
Hopefully you see that the column under $\neg P \lor Q$ and $P \rightarrow Q$ are the same! This tells us that the two statements are equivalent since they have the same truth values. (From before, we know that one can also rewrite $\neg P \lor Q$ to find other equivalencies.)

## 5.1 Contrapositive versus converse

Contrapositive and converse are not the same thing. The converse of the statement $P \rightarrow Q$ is $Q \rightarrow P$. They are not equivalent. (You can confirm this by checking their truth tables.) You can also see this intuitively. The statement "If you have a PhD then you are smart" is different from "If you are smart then you have a PhD".

The contrapositive of $P \rightarrow Q$ is $\neg Q \rightarrow \neg P$. The two statements are equivalent. You can verify this with a truth table. Again, we can see this intuitively. The statement "If you live in Albuquerque, then you live in New Mexico" is the same as "If you don’t live in New Mexico then you don’t live in Albuquerque".

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**Exercise 28** Which of the following statements are equivalent?

1. If it’s either raining or snowing, then the game has been canceled.

2. If the game hasn’t been canceled, then it’s not raining and it’s not snowing.
3. If the game has been canceled then it’s either raining or snowing.

It turns out that there are a number of different ways of saying \( P \rightarrow Q \).

- If \( P \) then \( Q \)
- \( P \) implies \( Q \)
- \( P \) only if \( Q \)
  
  Consider the sentence “You can vote in US national elections only if you are a US citizen”. In other words, if you aren’t a US citizen, you can’t vote in US national elections. This is \( \neg Q \rightarrow \neg P \) which we know is equivalent to \( P \rightarrow Q \).

- \( P \) is a sufficient condition for \( Q \)
- \( Q \) is a necessary condition for \( P \)
  
  This means that if \( Q \) doesn’t occur than \( P \) can’t occur. In other words, \( \neg Q \rightarrow \neg P \) or \( P \rightarrow Q \).

Exercise 29

Analyze the logical forms of the following sentences

1. If at least 10 people are there, then the lecture will be given

2. The lecture will be given only if at least 10 people are there
3. The lecture will be given if at least 10 people are there.

4. Having at least 10 people there is a sufficient condition for the lecture being given.

5. Having at least 10 people there is a necessary condition for the lecture being given.

5.2 A last bit of notation

$P \iff Q$ is a biconditional statement and it is shorthand for $(P \rightarrow Q) \land (Q \rightarrow P)$. This is translated as "$P$ if $Q$ and $P$ only if $Q$". More often you’ll see this expressed as ”if and only if” or iff. Later on, we’ll often see this expressed as ”$P$ is a necessary and sufficient condition for $Q$”.

Exercise 30 Analyze the logical forms of the following sentences.

1. The game will be canceled iff it’s either raining or snowing.

2. Having at least 10 people there is a necessary and sufficient condition for the lecture being given.
3. If John went to the store then we have some eggs, and if he didn’t then we don’t.

6 Quantificational Logic

6.1 Quantifiers

A little more notation:

- $\forall$ means "for all"
  - ex) $\forall x P(x)$ means that "for all values of $x$, $P(x)$ is true"

- $\exists$ means "there exists"
  - ex) $\exists x P(x)$ means that "there exists a value of $x$, such that $P(x)$ is true"

Words like everyone, someone, everything, or something are signals that you will need a quantifier.

Exercise 31 What do the following mean? Are they true or false?

1. $\forall x (x^2 \geq 0)$, where the universe is $\mathbb{R}$

2. $\exists x (x^2 - 2x + 3 = 0)$, where the universe is $\mathbb{R}$

3. $\exists x (M(x) \land B(x))$, where the universe is the set of all people, $M(x)$ stands for the statement "$x$ is a man" and $B(x)$ means "$x$ has brown hair".
4. $\forall x (M(x) \to B(x))$ with the same universe and meanings as above

**Exercise 32** What do the following mean? Are they true or false? The universe is $\mathbb{N}$, natural numbers.

1. $\forall x \exists y (x < y)$

2. $\exists y \forall x (x < y)$

3. $\exists x \forall y (x < y)$

Note: all mathematical statements can be understood using the 7 logical symbols that we have so far introduced: $\lor$, $\land$, $\neg$, $\rightarrow$, $\leftrightarrow$, $\exists$, $\forall$. 