Representation Theory of a Graph

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Abstract
In this work, we study a generalization of the concept of modules by considering more general setting which is called OG-diagrams. We use Higman criteria and relative projectivity as the main tools to get our results.

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1 Preliminaries
Module theory is one modern approach to study algebras. Our main concern is to use graph theory to generalize some module theory in the present of special graph( Chapter 1, Section 5 in [2]) which we can define as follows:
Definition 1.1 A finite oriented graph is a triple \((D, E, \mu)\), where \(D\) and \(E\) are finite sets, the elements of \(D\) are called vertices and the elements of \(E\) are called edges. For all \(e \in E\) the map \(\mu : E \rightarrow D \times D\) is defined by the rule \(\mu(e) = (d_1, d_2)\) where \(d_1\) is called the origin of \(e\) and \(d_2\) is called the extremity of \(e\).

In fact, many edges have the same origin and extremity and we identify a graph with its set \(D\) of vertices. In this paper all groups are finite and all algebras are finite dimension over an algebraically closed field of characteristic \(p > 0\). Now consider the collections \(\text{Mod} = \{M_d : d \in D\}\) where \(M_d\) is a finitely generated \(OG\)-module and the collection \(\text{Map} = \{f_e : e \in E\}\) where \(f_e : M_{d_1} \rightarrow M_{d_2}\) and \(f_e\) is an \(OG\)-map.

Definition 1.2 An \(OG\)-diagram with respect to an oriented graph \(D\) is a pair \((M, f)\) where \(M\) is a function from \(D\) to the collection of \(OG\)-modules \(\text{Mod}\) by taking \(d \in D\) on \(M_d \in \text{Mod}\) and similarly \(f\) is a function from \(E\) to the collection of \(OG\)-linear maps \(\text{Map}\) by taking \(e \in E\) on \(f_e \in \text{Map}\).

By Definition 1.2, an \(OG\)-module is an \(OG\)-diagram with respect to an oriented graph with a single vertex and no edge. So, our aim is to generalize some result from module theory to \(OG\)-diagram concept. The main tools are the so-called Krull-Shmidt Theorem and Higman criterion. We shall follow Chapter 5 in [1].

Now consider two \(OG\)-diagrams \((M, f)\) and \((\hat{M}, \hat{f})\) of the graph \(D\) then an \(OG\)-linear homomorphism \(\psi : (M, f) \rightarrow (\hat{M}, \hat{f})\) is a family of \(OG\)-linear maps \(\psi_d : M_d \rightarrow \hat{M}_d\), indexed by the set \(D\) of vertices such that for every edge \(e\) with origin \(d_1\) and extremity \(d_2\), we have \(\hat{f}_e \circ \psi_{d_1} = \psi_{d_2} \circ f_e\).

We get the \(O\)-module \(\text{Hom}_{OG}((M, f), (\hat{M}, \hat{f}))\) which consists of all \(OG\)-linear homomorphisms from \((M, f)\) to \((\hat{M}, \hat{f})\). In the case that \((M, f) = (\hat{M}, \hat{f})\) we get the endomorphism algebra \(\text{End}_{OG}(M, f)\) of all \(OG\)-linear endomorphisms of \((M, f)\). For a fixed oriented graph \(D\), the \(OG\)-diagrams of \(D\) together with the \(OG\)-linear homomorphisms form an abelian category. Hence it is appropriate to introduce the concepts of restriction and induction functors on this category.

Definition 1.3 Let \(H\) be a subgroup of \(G\). The restriction functor \(\text{Res}_H^G : ((M, f), \psi) \rightarrow (\text{Res}_H^G(M, f), \text{Res}_H^G(\psi))\) which sends the \(OG\)-diagram \((M, f)\) to
the OH-diagram \((\text{Res}_H^G(M, f))\) of the same graph and sends the OG-linear homomorphism \(\psi\) to the OH-linear homomorphism \(\text{Res}_H^G(\psi)\) where \(\text{Res}_H^G(M_d) = \text{Res}_H^G(M_d)\) and \(\text{Res}_H^G(f_e) = \text{Res}_H^G(f_e)\) for every vertex \(d\) and for every edge \(e\).

**Definition 1.4** Let \((M, f)\) be an OH-diagram of shape \(D\) where \(H\) is a subgroup of \(G\), we can define the induction functor \(\text{Ind}_H^G : ((M, f), \psi) \rightarrow (\text{Ind}_H^G(M, f), \text{Ind}_H^G(\psi))\) which sends the OH-diagram \((M, f)\) to the OG-diagram \((\text{Ind}_H^G(M, f))\) of the same graph and sends the OH-linear homomorphism \(\psi\) to the OG-linear homomorphism \(\text{Ind}_H^G(\psi)\) where \(\text{Ind}_H^G(M_d) = \text{Ind}_H^G(M_d)\) and \(\text{Ind}_H^G(f_e) = \text{Ind}_H^G(f_e)\) for every vertex \(d\) and for every edge \(e\).

**Definition 1.5** For a fixed finite oriented graph \(D\), the direct sum of two OG-diagrams \((M, f)\) and \((\hat{M}, \hat{f})\) is the OG-diagram \((M \oplus \hat{M}, f \oplus \hat{f})\) where \((M \oplus \hat{M})_d = M_d \oplus \hat{M}_d\) for every vertex \(d\) and \((f \oplus \hat{f})_e = f_e \oplus \hat{f}_e\) for every edge \(e\).

**Definition 1.6** An OG-diagram is called indecomposable if it is not zero and if it cannot be decomposed as a direct sum of non-zero OG-diagrams.

For the OG-diagram \((M, f)\) of a fixed shape \(D\) where \(O\) is an algebraically closed field of characteristic \(p > 0\) we can define the endomorphism algebra \(\text{End}_O(M, f)\) which is a \(G\)-algebra over \(O\) and consists of all \(O\)-linear endomorphisms of \(\text{Res}_1^G(M, f)\). This \(G\)-algebra contains \(\text{End}_{OG}(M, f)\) as a subalgebra. Moreover it is an interior \(G\)-algebra. We record this fact in the following lemma:

**Lemma 1.7** If \(H\) is a subgroup of \(G\) and \((M, f)\) is an OG-diagram then the endomorphism algebra \(\text{End}_O(M, f)\) is an interior \(G\)-algebra.

Proof:
The proof can be seen in [2] page 256.

**Lemma 1.8** Suppose that \(D\) be a finite oriented graph and \(H \leq G\). Let \((M, f)\) be an OH-diagram of the shape \(D\) then there is an isomorphism of interior \(G\)-algebras \(\text{End}_O(\text{Ind}_H^G(M, f)) \cong \text{Ind}_H^G(\text{End}_O(M, f))\).

Proof:
The proof can be seen in [2] page 254.

**Lemma 1.9** If \(H \leq G\), then \(\text{End}_O(M, f)^H = \text{End}_{OH}(M, f)\) where \(\text{End}_O(M, f)^H\) is the endomorphism algebra of OG-linear homomorphisms which are fixed under \(H\).
Proof:
Since the element \( \psi \in \text{End}_O(M,f) \) is fixed under \( H \) if and only if each component \( \psi_d \) is fixed under \( H \), which means that \( \psi_d \in \text{End}_{OH}(M_d) \). That is \( \psi \in \text{End}_{OH}(M,f) \).

**Lemma 1.10** An \( OG \)-diagram \((M,f)\) is indecomposable if and only if the endomorphism algebra \( \text{End}_{OG}(M,f) \) is a local algebra over \( O \).

Proof:
Suppose that \((M,f)\) is an indecomposable \( OG \)-diagram, from Definition 1.6 it is not zero and it cannot be decomposed as a direct sum of non-zero \( OG \)-diagrams. So there is no idempotent in \( \text{End}_{OG}(M,f) \) except 0 and 1.
On the other hand, Let \((M,f)\) be an indecomposable \( OG \)-diagram and \( i \in \text{End}_{OG}(M,f) \) be an idempotent, then the image of \( i \) will be a direct summand of \((M,f)\), hence \((M,f)\) is indecomposable.

**Lemma 1.11** For a fixed finite oriented graph \( D \). There exists a bijection between the decomposition of the \( OG \)-diagram \((M,f)\) as a direct sum of an indecomposable \( OG \)-diagrams and the decomposition of the identity element \( \text{id}_{(M,f)} \) as primitive idempotents of \( \text{End}_{OG}(M,f) \).

Proof:
The proof can be seen in [2] page 255.

**Corollary 1.12** Any direct summand of the \( OG \)-diagram \((M,f)\) is indecomposable if and only if the correspondence idempotent is primitive idempotent of \( \text{End}_{OG}(M,f) \).

The following Lemma is the Krull-Shmidt Theorem holds for the \( OG \)-diagrams:

**Lemma 1.13** Let \( D \) be a finite oriented graph and let \((M,f)\) be an \( OG \)-diagram of shape \( D \). There exists a unique decomposition \((M,f) = \bigoplus_{i=1}^{n}(M_i,f_i)\) as a finite direct sum of indecomposable \( OG \)-diagrams.

Proof:
The proof can be seen in [2] page 255.

### 2 Defect group of the \( OG \)-diagram

Through this section we shall study the concept of relative projectivity for the \( OG \)-diagram:
Definition 2.1 Let $H \leq G$. Then the OG-diagram $(M, f)$ is said to be projective relative to $H$ if there is an OH-diagram $(N, g)$ such that $(M, f)$ is isomorphic to a direct summand of $\text{Ind}^G_H(N, g)$. As a notation, we write $(M, f) \mid \text{Ind}^G_H(N, g)$

Now for $H \leq G$ and $x \in G$, we consider that $H^x = xHx^{-1} = \{xhx^{-1}, h \in H\}$. We need to define the $G$-conjugate of an OH-diagram $(N, g)$ with shape $D$ as follows:

Definition 2.2 For $x \in G$, the $G$-conjugate of an OH-diagram $(N, g)$ which denoted by $(N, g)^x$ is an OH$^x$-diagram such that $(N, g)^x = (N^x, g^x)$ where $N^x = \{n_d^x : x \in G, d \in D\}$ is the OH$^x$-module and $g^x$ is the OH$^x$-linear maps for all $e \in E$. Note that $N^x$ is a function from $D$ to the collection of $\text{Mod}N^x_d$ and $g^x$ is a function from $E$ to the collection $g^x_e$ where $g^x_e : N^x_d \to N^x_e$

Definition 2.3 For $H \leq G$, the $H$-invariant of an OG-diagram $(M, f)$ which is denoted by $\text{Inv}_H((M, f))$ is defined as $(\text{Inv}_H(M_d), \text{Inv}_H(f_e))$ where

\[
\text{Inv}_H(M_d) = \{m_d \in M_d : m^h_d = m_d\}
\]

and

\[
\text{Inv}_H(f_e) = \{f_e : f^h_e = f_e\}
\]

for all $h \in H, d \in D, e \in E$

Remark:

Notice that if $H_1 \leq H_2 \leq G$ then $\text{Inv}_{H_2}((M, f)) \subset \text{Inv}_{H_1}((M, f))$ and in particular, $\text{Inv}_G((M, f)) \subset \text{Inv}_{H_1}((M, f))$.

Lemma 2.4 For a fixed finite oriented graph $D$, and let $H \leq G$ be an OG-diagram then $\text{Inv}_{H^x}((M, f)) = (\text{Inv}_H((M, f)))^x$

Proof:

Let $(m_d, f_e) \in \text{Inv}_{H^x}((M, f))$. That means $(m_d, f_e).h^x = (m_d, f_e)$ for all $h \in H$. Hence $(m_d, f_e).h^x = (m_d, f_e).x^{-1}hx = (m_d, f_e)x^{-1}h = (m_d, f_e).x^{-1}$. Which means that $(m_d, f_e) \in \text{Inv}_H((M, f))$.

On the other hand, if we let $(m_d, f_e) \in \text{Inv}_H((M, f)).x$. Then $(m_d, f_e) = (m_d, f_e).x$, where $(m_d, f_e) \in \text{Inv}_H((M, f))$. That is $(m_d, f_e).h^x = (((m_d, f_e))).x.h^x = (((m_d, f_e))).x.x^{-1}hx = (((m_d, f_e))).hx = (((m_d, f_e))).x$. Hence $(m_d, f_e).h^x = (m_d, f_e)$ implies $(m_d, f_e) \in \text{Inv}_{H^x}((M, f))$

We need to define the relative trace map in the case of OG-diagram as follows:
Definition 2.5 Let $H \leq G$ and $T$ a set of coset representatives of $H$ in $G$. Consider an OG-diagram $(M, f)$ with respect to a finite oriented graph $D$. We can define the relative trace map

$$Tr^G_H : \text{Inv}_H((M, f)) \to \text{Inv}_G((M, f))$$

by:

$$Tr^G_H((m_d, f_e)) = \sum_{t \in T} (m_d, f_e)^t$$

for all $d \in D$ and $e \in E$.

The following theorem is a generalization of a fundamental results in Representation theory (See Theorem 2.2 in [1]):

Theorem 2.6 Let $H \leq G$ and $(M, f)$ be an OG-diagram. The following conditions are equivalent:

1. The OG-diagram $(M, f)$ is projective relative to $H$.
2. There exists $\theta \in \text{End}_{OH}(M, f)$ such that $Tr^G_H(\theta) = \text{id}_{(M, f)}$.
3. $\text{End}_{OG}(M, f) = Tr^G_H(\text{End}_O((M, f)))$.
4. For the following figure of OG-diagrams and OG-homomorphisms $\eta$ and $\mu$ with the exact row

\[
\begin{array}{c}
\phi \\
\downarrow \eta \\
(L, k) \xrightarrow{\mu} (N, g) \xrightarrow{} 0
\end{array}
\]

there exists an OG-homomorphisms $\varphi : (M, f) \to (L, k)$ such that $\eta = \mu \circ \varphi$, provided there exists an OH-homomorphisms $\psi : (M, f) \to (L, k)$ such that $\eta = \mu \circ \psi$.
5. Every epimorphism $\mu : (L, k) \mapsto (M, f)$ of OG-diagrams splits if it splits as an OH-homomorphism.
6. The OG-epimorphism $\lambda : \text{Ind}_H^G(\text{Res}_H^G(M, f)) \mapsto (M, f)$ splits.
7. The OG-diagram $(M, f)$ is a direct summand of $\text{Ind}_H^G(\text{Res}_H^G(M, f))$. 
Proof:

(1) $\implies$ (2):

Suppose that $(M, f)$ is an OG-diagram projective relative to $H$, that means $\text{Ind}_H^G(N, g) = (M, f) \oplus (\hat{M}, \hat{f})$ for some $OH$-diagram $(N, g)$ and $OG$-diagram $(\hat{M}, \hat{f})$. By Definition of the induction functor we have

$$\text{Ind}_H^G(N, g) = \bigoplus_{t \in [H \setminus G], t \neq 1} (N, g) \otimes t$$

that is

$$\text{Ind}_H^G(N, g) = ((N, g) \otimes 1) \oplus \bigoplus_{t \in [H \setminus G], t \neq 1} (N, g) \otimes t$$

Now let $\varepsilon \in \text{End}_{OH}(\text{Ind}_H^G(N, g))$ be the projection on $(M, f)$ and $\tau \in \text{End}_{OH}(\text{Ind}_H^G(N, g))$ be the projection on $(N, g) \otimes 1$ in the previous $OH$-decomposition of $\text{Ind}_H^G(N, g)$.

Then we have $\text{Tr}_H^G(\tau) = \text{id}_{(\text{Ind}_H^G(N, g))}$ and by Lemma 1.6 in [1] we have $\varepsilon \text{Tr}_H^G(\tau) \varepsilon = \varepsilon \text{Tr}_H^G(\varepsilon \varepsilon) = \varepsilon \text{id}_{(\text{Ind}_H^G(N, g))} \varepsilon$. Therefore we get $\text{Tr}_H^G(\varepsilon) = \text{id}_{(M, f)}$, where $\varepsilon = \varepsilon \varepsilon |_{(M, f)} \in \text{End}_{OH}((M, f))$.

(2) $\iff$ (3):

By hypothesis let $\theta \in \text{End}_{OH}(M, f)$ such that $\text{Tr}_H^G(\theta) = \text{id}_{(M, f)}$. Since $\text{Tr}_H^G(\text{End}_{OH}(M, f))$ is an ideal of $\text{End}_{OG}(M, f)$. Therefore $\text{End}_{OG}(M, f) = \text{Tr}_H^G(\text{End}_{OH}(M, f))$.

(2) $\implies$ (4):

suppose that $\theta \in \text{End}_{OH}(M, f)$ such that $\text{Tr}_H^G(\theta) = \text{id}_{(M, f)}$. Then $\mu \circ \text{Tr}_H^G(\psi \circ \theta) = \text{Tr}_H^G(\mu \circ \psi \circ \theta) = \text{Tr}_H^G(\eta \circ \theta) = \eta \circ \text{Tr}_H^G(\theta) = \eta$. Thus it suffices to set $\varphi = \text{Tr}_H^G(\psi \circ \theta)$.

(4) $\implies$ (5):

By hypothesis and if we put $(M, f) = (N, g)$ and $\eta = \text{id}_{(M, f)}$ in (4) we get $\text{id}_{(M, f)} = \mu \circ \psi$ which means that $\mu$ is split.

(5) $\implies$ (6):

Since the $\pi : (M, f) \to \text{Ind}_H^G(\text{Res}_H^G(M, f))$ which defines by $\pi((m, f)) = (m, f) \otimes 1$ is an $OH$-homomorphism such that $\lambda \circ \pi = \text{id}_{(M, f)}$. Hence, from the assumption, $\lambda$ splits as an $OG$-homomorphism.

(6) $\implies$ (7):

Suppose that the $OG$-epimorphism $\lambda : \text{Ind}_H^G(\text{Res}_H^G(M, f)) \twoheadrightarrow (M, f)$ is split, that means there exist $OG$-homomorphism $\eta : (M, f) \Rightarrow \text{Ind}_H^G(\text{Res}_H^G(M, f))$ such that $\eta \circ \lambda = 1_{\text{Ind}_H^G(\text{Res}_H^G(M, f))}$. Hence the kernel $\text{Ker}\lambda$ be direct summand of $\text{Ind}_H^G(\text{Res}_H^G(M, f))$, but $\text{Ker}\lambda = (M, f)$. Therefore the result follows.

(7) $\implies$ (1):

By hypothesis the $OG$-diagram $(M, f)$ is a direct summand of $\text{Ind}_H^G(\text{Res}_H^G(M, f))$ and from Definition 2.1 we have that the $OG$-diagram $(M, f)$ is projective relative to $H$. 

Remark:
Condition (2) in Theorem 2.6 is refereed to as Higman’s criterion.
In fact the three following lemmas gives us an interesting results about the
properties of relative projectivity of $OG$-diagrams:

**Lemma 2.7** Let the $OG$-diagram $(M, f)$ be projective relative to $H$ and $H \leq T \leq G$ then $(M, f)$ is projective relative to $T$.

**Proof:**
Since the $OG$-diagram $(M, f)$ is projective relative to $H$ then $(M, f)$ isomorphic to a direct summand of $Ind^G_H(N, g)$ for some $OH$-diagram $(N, g)$ but by the Definition of the induction of the $OH$-diagrams 1.4 we have that

$$Ind^G_H(N, g) = (Ind^G_H N_d, Ind^G_H g_e) = (Ind^G_T N_d, Ind^G_T g_e) = Ind^G_T(N, g)$$
so the result follows directly.

**Lemma 2.8** Let the $OG$-diagram $(M, f)$ be projective relative to $H$ and $x \in G$ then $(M, f)$ be projective relative to $H^x$.

**Proof:**
Since the $OG$-diagram $(M, f)$ is projective relative to $H$ then $(M, f)$ isomorphic to a direct summand of $Ind^G_H(N, g)$ for some $OH$-diagram $(N, g)$ but by Definition 2.5 we have that $(N, g)^x$ is an $OH^x$-diagram and since $Ind^G_H(N, g)^x \simeq Ind^G_H(N, g)$ the result follows.

**Definition 2.9** If $A = End_O(M, f)$ then the defect group of $(M, f)$ is the defect group of $1_A$.

An indecomposable $OG$-diagram $(M, f)$ has a defect group $Q$ if and only if $Q$ is a defect group of the primitive algebra $End_{OG}(M, f)$.

**Proposition 2.10** If the indecomposable $OG$-diagram $(M, f)$ is projective relative to $H$ then $M_d$ is projective relative to $H$ for all $d \in D$.

**Proof:**
Suppose that $(M, f)$ is projective relative to $H$. That is $(M, f)$ is isomorphic to a direct summand of $Ind^G_H(N, g)$ for some $OH$-diagram $(N, g)$. But by Definition 1.4 we have $(M, f)$ isomorphic to a direct summand of $(Ind^G_H N_d, Ind^G_H g_e)$. So we find that the $OG$- module $M_d$ is isomorphic to a direct summand of $Ind^G_H(N_d)$ for some $OH$-modules $N_d$. Therefore $M_d$ is projective relative to $H$ for all $d \in D$.

By Higman Criterion and the previous proposition we can proof the following result:
Corollary 2.11 The vertex of usual module $M_d$ is contained in the defect group of the OG-diagram $(M, f)$, for each $d \in D$.

Proof:
Suppose that $Q$ be a defect group of the OG-diagram $(M, f)$. Then by Proposition 2.10 we have that $M_d$ is projective relative to $Q$ for all $d \in D$. Hence the vertex of $M_d$ is contained in $Q$ as the vertex is minimal subgroup with respect to relative projectivity.

Theorem 2.12 If $(M, f)$ is an OG-diagram and $H \leq G$ such that $[G : H]$ is prime to $p$, then $(M, f)$ is $H$-projective.

Proof:
Since $[G : H]$ is prime to $p$ then $[G : H] \cdot 1_O \neq 0$. So that $[G : H]$ is invertible in $O$. Hence $\text{Tr}_H^G([G : H])^{-1} \cdot 1_{(M,f)_H} = ([G : H])^{-1} \cdot 1_{(M,f)} = 1_{(M,f)}$.

Corollary 2.13 Each OG-diagram $(M, f)$ is $H$-projective for each sylow $p$-subgroup $H$ of $G$.

Proof:
Follows directly from Theorem 2.12.

References


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